Example in Integral Calculus.
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S. B. Moon.

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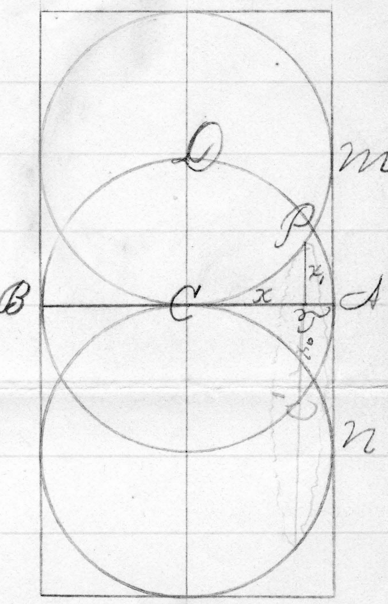

Let ADB be the directrix, and MN a diameter of the generatrix in its starting position. The volume generated in going from A to B is $\frac{V'}{2} = 2\pi a^2\int_0^a dx = 2\pi a^3$. In returning to A, the circle generates an equal volume but part of it is common to that generated in going from A to B.

Hence $V' = 4\pi a^3$, and it remains to find the common portion and subtract it from V'. A section of this portion, perpendicular to AC, gives two segments of circles, radius $= a$, having their chord in common. From the Differential Calculus, area of segment $= \frac{\pi a^2}{2} - x_0\sqrt{a^2 - x_0^2} - a^2\sin^{-1}\frac{x_0}{a}$, where x_0 is measured from the centre. We want this area in terms of $a - x_0 = x_1 = PT$, and it becomes $\frac{\pi a^2}{2} - (a - x_1)\sqrt{2ax_1 - x_1^2} - a^2\sin^{-1}\frac{a - x_1}{a}$. But $x^2 = \overline{CT}^2$ ⊕ $= 2ax_1 - x_1^2$, and $dx = \frac{(a - x_1)\,dx_1}{\sqrt{2ax_1 - x_1^2}}$.

$$\frac{1}{4}V'' = \int_0^a \left\{\frac{\pi a^2}{2} - (a - x_1)\sqrt{2ax_1 - x_1^2} - a^2\sin^{-1}\frac{a - x_1}{a}\right\}dx$$

$$= \int_0^a \left\{\frac{\pi a^2}{2} - (a - x_1)\sqrt{2ax_1 - x_1^2} - a^2\sin^{-1}\frac{a - x_1}{a}\right\}\frac{a - x_1}{\sqrt{2ax_1 - x_1^2}}\,dx_1$$

$$= \int_0^a \frac{\pi a^2}{2}\cdot\frac{a - x_1}{\sqrt{2ax_1 - x_1^2}}\,dx_1 - \int_0^a (a - x_1)^2\,dx_1 - a^2\int_0^a \frac{a - x_1}{\sqrt{2ax_1 - x_1^2}}\cdot\sin^{-1}\frac{a - x_1}{a}\,dx_1$$

$$= \frac{\pi a^2}{2}\sqrt{2ax_1 - x_1^2} + \frac{(a - x_1)^3}{3} - a^2\int u\,dv \qquad \left\{\begin{array}{l} u = \sin^{-1}\frac{a - x_1}{a},\quad du = -\frac{dx_1}{\sqrt{2ax_1 - x_1^2}} \\ dv = \frac{a - x_1}{\sqrt{2ax_1 - x_1^2}}\,dx_1,\quad v = \sqrt{2ax_1 - x_1^2} \end{array}\right.$$

$$= \frac{\pi a^2}{2}\sqrt{2ax_1 - x_1^2} + \frac{(a - x_1)^3}{3} - a^2\sqrt{2ax_1 - x_1^2}\left(\sin^{-1}\frac{a - x_1}{a}\right) - a^2\int dx_1$$

$$\frac{1}{4}V'' = \frac{\pi a^3}{2} - \frac{a^3}{3} - a^3 = \frac{\pi a^3}{2} - \frac{4}{3}a^3, \quad V'' = 2\pi a^3 - \frac{16}{3}a^3$$

$$V' - V'' = 4\pi a^3 - 2\pi a^3 + \frac{16}{3}a^3 = \frac{2a^3}{3}(3\pi + 8) = \text{Ans.}$$

The first term of the integral might as well have been taken $\int_0^a \frac{\pi a^2}{2}dx$ as $\int_0^a \frac{\pi a^2}{2}\cdot\frac{a - x_1}{\sqrt{2ax_1 - x_1^2}}\,dx_1$.

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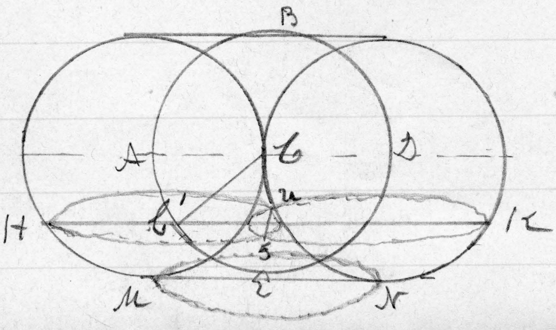
⊕ Since CT = half the chord of the segments, for if the circle of which the segment is a part were revolved on its diameter the circumf wd pass thro' C.



Let I be the middle of the chord SU

Let ABDE be the directrix. MN a diameter of the generatrix in its first position.

The volume generated in passing from E to B is $2\pi a^3$
In returning to its first position it generates also $2\pi a^3$
Making the entire volume $V' = 4\pi a^3$.
But part of this volume is repeated. We must therefore find the common portion and subtract it from V'

A section of this portion made by a plane HK perp. to BE gives two equal segments of circles having a common chord SU. From Byerly's Diff'l Calculus we find the area of one segment of a circle $= \frac{\pi a^2}{2} - x_0\sqrt{a^2-x_0^2} - a^2\sin^{-1}\frac{x_0}{a}$
where x_0 = distance from C' to I. But from triangle C'CI
$x_0 = \sqrt{a^2-x^2}$ where $x = CI$.
Hence area of segment $= \frac{\pi a^2}{2} - x\sqrt{a^2-x^2} - a^2\sin^{-1}\frac{\sqrt{a^2-x^2}}{a}$

$\therefore \frac{1}{4}$ vol repeated $= \frac{1}{4}V'' = \int \left[\frac{\pi a^2}{2} - x\sqrt{a^2-x^2} - a^2\sin^{-1}\frac{\sqrt{a^2-x^2}}{a}\right]_0^a dx$

$\frac{1}{4}V'' = \frac{\pi a^2 x}{2} + \frac{1}{3}(a^2-x^2)^{3/2} - a^2\int \sin^{-1}\frac{\sqrt{a^2-x^2}}{a}\,dx$

Integrating the last term by parts

$$\frac{1}{4}V'' = \left[\frac{\pi a^2 x}{2} + \frac{1}{3}(a^2-x^2)^{3/2} - a^2 x\sin^{-1}\frac{\sqrt{a^2-x^2}}{a} + a^2(a^2-x^2)^{\frac{1}{2}}\right]_0^a$$

$$\frac{1}{4}V'' = \frac{\pi a^3}{2} - \frac{4}{3}a^3$$

$$V'' = 2\pi a^3 - \frac{16}{3}a^3$$

& $V' - V'' = 2\pi a^3 + \frac{16}{3}a^3 = \frac{2a^3}{3}(3\pi + 8)$