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Prop. To determine the equation of the tangent-plane to a given surface at a given point $P_1 (x_1 y_1 z_1)$.

The general equation of a plane is

$$Ax + By + Cz + D = 0$$

If it pass through the point $P(x_1 y_1 z_1)$ we have

$$A(x-x_1) + B(y-y_1) + C(z-z_1) = 0 \qquad (1)$$

We may determine the values of A B ~~&~~ C by imposing conditions of tangency.

~~Suppose (1) to be a tangent plane at P_1~~

Let the surface & plane be intersected by secant planes passing through P & parallel respectively to the planes of xz & yz.

The equations of the line cut from the tangent plane by the plane parallel to xz will be

$$x - x_1 = t(z-z_1) \quad (2) \quad \& \quad y = y_1 \quad (3)$$

and those of the line cut by the plane parallel to yz

$$y - y_1 = s(z-z_1) \quad (4) \quad \& \quad x = x_1$$

The tangent plane will contain these lines.

The trace of (1) on xz is

$$A(x-x_1) - By_1 + C(z-z_1) = 0 \qquad (6)$$

And that on yz

$$-Ax_1 + B(y-y_1) + C(z-z_1) = 0 \qquad (7)$$

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But (6) is parallel to (2) & (7) to (4)

$\therefore\ t = -\frac{C}{A}$ & $S = -\frac{C}{B}$ & (1) becomes

$$z - z_1 = \frac{1}{t}(x - x_1) + \frac{1}{s}(y - y_1) \qquad (8)$$

Since (2) (3) & (4) (5) are respectively tangent to to the corresponding curves cut from the surface we must have $t = \frac{dx_1}{\partial z_1}$ & $S = \frac{dy_1}{\partial z_1}$

or $\frac{1}{t} = \frac{\partial z_1}{\partial x_1}$ & $\frac{1}{S} = \frac{\partial z_1}{\partial y_1}$ & (8) becomes

$$z - z_1 = \frac{dz_1}{\partial x_1}(x - x_1) + \frac{dz_1}{\partial y_1}(y - y_1) \qquad (9)$$

Where $\frac{\partial z_1}{\partial x_1}$ & $\frac{\partial z_1}{\partial y_1}$ are the partial differential coefficients derived from the equation of the surface, and they will have the same values at the point P. as the similar coefficients derived from the equation of the plane tangent at that point.

If the equation of the surface be $u = \varphi(x, y, z) = 0$

$\left[\frac{du}{\partial x}\right] = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0$ & $\left[\frac{du}{\partial y}\right] = \frac{du}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = 0$ and

$$\frac{dz_1}{\partial x_1} = -\frac{\frac{du}{\partial x_1}}{\frac{du}{\partial z_1}}, \quad \frac{dz_1}{\partial y_1} = -\frac{\frac{du}{\partial y_1}}{\frac{du}{\partial z_1}}$$

Substituting in (9) & we have

$$(x - x_1)\frac{du}{\partial x_1} + (y - y_1)\frac{du}{\partial y_1} + (z - z_1)\frac{du}{\partial z_1} = 0$$

or as in Byerly Vol II page 161.

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A line normal to the surface at P. will have for its equations $x - x_1 = t'(z - z_1)$ $y - y_1 = s'(z - z_1)$

And since the projections of the normal are perpendicular to the traces of the tangent plane

$A = Ct'$ & $B = Cs'$

$\therefore\ t' = \frac{A}{C} = -\frac{dz_1}{dx_1}$, $s' = \frac{B}{C} = -\frac{dz_1}{dy_1}$

If a line be drawn through the origin perpendicular to the tangent plane making the angles $\alpha\ \beta\ \gamma$ with the axes of $x\ y$ & z, it will be parallel to the normal at P.

In notes on Analytical Geometry in Space, we have

$$\cos X = \frac{a}{\sqrt{1+a^2+b^2}},\ \cos Y = \frac{b}{\sqrt{1+a^2+b^2}},\ \cos Z = \frac{1}{\sqrt{1+a^2+b^2}}$$

Exchanging X for α, Y for β, Z for γ a for t' & b for s' we have.

$$\cos\alpha = \frac{-\frac{dz_1}{dx_1}}{\sqrt{1+\frac{dz_1^2}{dx_1^2}+\frac{dz_1^2}{dy_1^2}}} = \frac{\frac{du}{dx_1}}{\sqrt{\frac{du^2}{dx_1^2}+\frac{du^2}{dy_1^2}+\frac{du^2}{dz_1^2}}}$$

$$\cos\beta = \frac{-\frac{dz_1}{dy_1}}{\sqrt{1+\frac{dz_1^2}{dx_1^2}+\frac{dz_1^2}{dy_1^2}}} = \frac{\frac{du}{dy_1}}{\sqrt{\frac{du^2}{dx_1^2}+\frac{du^2}{dy_1^2}+\frac{du^2}{dz_1^2}}}$$

As in Byerly.

$$\cos\gamma = \frac{1}{\sqrt{1+\frac{dz_1^2}{dx_1^2}+\frac{dz_1^2}{dy_1^2}}} = \frac{\frac{du}{dz_1}}{\sqrt{\frac{du^2}{dx_1^2}+\frac{du^2}{dy_1^2}+\frac{du^2}{dz_1^2}}}$$