Realizability of n-Vertex Graphs with Prescribed Vertex Connectivity, Edge Connectivity, Minimum Degree, and Maximum Degree

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1 Introduction

The study of graph theory has a long tradition in mathematics. With the publication of the *Seven Bridges of Königsberg* in 1736, L. Euler proved that it was impossible to devise a walk that started and ended at the same place while crossing the seven bridges exactly one time. His discovery marked the beginning of graph theory and its endless applications. It followed that mathematicians such as A.L. Cauchy (1789-1857) and A. Cayley (1821-1895) utilized graphs to solve problems in other areas of mathematics. In an effort to study the intrinsic properties of these graphs, scientists and mathematicians looked for ways to describe the connectivity of a graph or network. In our research we continue the study of graph connectivity, continuing on ideas from earlier research by K. Menger (1902-1985), F. Harary (1921-2005), and most specifically F. T. Boesch and C. L. Suffel, [4] and [5], who realized graphs with given connectivity parameters. We will begin with a few definitions.

Definition 1. *A* ***graph*** *is a set of vertices and edges, denoted $G = (V, E)$. The vertices are a specific set $V = \{v_1, \dots, V_n\}$, while the edges are 2-element subsets of V, $E = \{e_1, \dots, e_m\}$ where each $e_k = v_iv_j$ for some $v_i, v_j \in V$.*

Definition 2. *The* ***degree*** *of a vertex is the number of edges that are incident, or connected, to it. The minimum degree of a graph G is denoted $\delta(G)$, and the maximum degree of the graph is denoted $\Delta(G)$.*

Definition 3. *A* ***loop*** *is an edge whose endpoints are the same vertex, $e_k \in E$ such that $e_k = v_iv_i$.*

Definition 4. *A* ***simple graph*** *is a graph with no loops or multiple edges between two vertices. In this study we only consider simple graphs.*

Definition 5. *A **path** is a trail from one vertex to another in which all vertices are distinct.*

Definition 6. *A **connected graph** is a graph where there is a path between any pair of vertices.*

Definition 7. *The **vertex connectivity** of a graph, G, is the smallest number of vertices that when removed disconnects the graph, denoted $\kappa(G)$. The **edge connectivity** is the smallest number of edges that when removed disconnects the graph, denoted $\lambda(G)$.*

Given a graph G, we can compute the values of κ, λ, δ, and Δ. This research investigates the converse: given a 4-tuple of these parameters, is there a graph G that realizes those specific values? This 4-tuple of parameters will be ordered appropriately by Whitney's theorem [7] that states

$$\kappa(G) \leq \lambda(G) \leq \delta(G) \leq \Delta(G). \tag{1}$$

In previous research W. Dymacek and other have completed a system of cases that realize a graph given any parameters and size. These realizations create graphs to satisfy the parameters, but are not unique. This research was started by L. Steiner working with W. Dymacek. In the first paper, A. Hardnett worked to realize parameters with $\kappa = 1$ and for $\kappa+\Delta \geq \lambda+\delta$. In the most recent publication, C. Bethea and W. Dymacek realized the 4-tuples in the form $(\kappa, \lambda, \delta, \Delta)$ where $\kappa + \Delta < \lambda + \delta$ and $\lambda < \delta$. With the compilation of all previous research, we study the final case of this project, $(\kappa, \delta, \delta, \Delta)$, where $\kappa > 1$ and the edge connectivity is equal to the minimum degree. Furthermore, in this case all parameters satisfy $\kappa + \Delta < \lambda + \delta$, and since we study $\lambda = \delta$,

$$\kappa + \Delta < 2\delta.$$

This paper will realize a given $(\kappa, \delta, \delta, \Delta)$ for any possible size, with $\kappa > 1$ where $\kappa + \Delta < 2\delta$.

2 Preliminaries

Given a 4-tuple of positive integers $(\kappa, \delta, \delta, \Delta)$, we begin with a realization function that produces the set of all orders for which we can realize a graph. This realization function, $F : \mathbb{N}^4 \to 2^{\mathbb{N}}$, is the set of all n for which there exists a graph of n vertices which realizes $(\kappa, \lambda, \delta, \Delta)$. We find that this realization function is not one-to-one. Before we can examine this function we will discuss common notation used to describe these realizations.

In this paper, we will refer to the number of vertices in the graph as the **order** of the graph. For simplicity $|G| = n$ will define this order, so that $|G| = |V|$.

Given two subgraphs H and G, we denote the set of connecting edges between H and G as $[H, G]$. For example, for a single $u \in H$ where $u \notin G$ where $G = \{s_1, s_2\}$, $[u, G]$ is the set of two edges us_1 and us_2. In our realizations we will use $\equiv$ to represent the set of connecting edges visually as $G \equiv H$. We will provide in depth descriptions of these edge sets. We will often let the size of $[H, G]$ also be denoted by $[H, G]$.

We let r_m be a binary variable depending on the parity of m. Hence, $r_m \in \{0, 1\}$ is given by

$$r_m = \begin{cases} 1 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

For example, if our parameters are $(4, 7, 7, 8)$, then $r_\delta = 1$ since δ is odd.

Definition 8. *A* ***complete graph*** *of size n represents a graph of n vertices, where each vertex is connected to every other vertex. We denote the complete graph of n vertices as K_n. This notation will commonly be used to describe subgraphs of our realizations. For K_n, there are n vertices and each vertex will have degree $n-1$.*

For non-negative integers k and n with $k < n$, the vertices of the **Harary graph**, $H_{n,k}$, are $V = \{v_0, v_1, \ldots, v_n - 1\}$ and for k even, the edges are $E = \left\{\{i, i \pm j\} : 0 \leq i < n, 0 < j \leq \frac{k}{2}\right\}$ where all arithmetic is done modulo n. If n is even and k is odd, to E we add the edges $\left\{\{v_i, v_{i+\frac{n}{2}}\} : 0 \leq i < \frac{n}{2}\right\}$. If nk is odd, then we add the following edges to E, $\left\{\{v_i, v_{i+\frac{n-1}{2}}\} : 0 \leq i < \frac{n}{2}\right\}$. The vertex and edge connectivity and the minimum degree of $H_{n,k}$ are k. The maximum degree of $H_{n,k}$ is $k + r_{nk}$ and if nk is odd, there is only one vertex of degree $k+1$, the others have degree k. Note that $H_{n,0} = N_n$, $H_{n,n-1} = K_n$, $H_{n,2} = C_n$, and $H_{n,1}$ is $\frac{n}{2}$ copies of K_2 if n is even and $\frac{n-3}{2}$ copies of K_2 and a P_3 if n is odd. We call the Harary graph irregular if nk is odd.

Definition 9. *For a* ***Harary graph*** *of order n, we define $H^\ell_{n,k}$ to be $H_{n,k}$ with an additional ℓ edges so that no vertex has degree larger than $k+1$. Thus $H^\ell_{n,k}$ has $2\ell + r_{nk}$ vertices of degree $k+1$ and the rest of degree k.*

To show our definition is well defined we note the following proof.

Proof. We can certainly add up to $\lfloor \frac{n}{2} \rfloor$ edges to $H_{n,k}$ if k is even for that is how we create $H_{n,k+1}$. If n is even and k is odd, we can add the edges $\left\{\{i, i + \frac{n-2}{2}\} : 0 \leq i < \frac{n-2}{2}\right\}$ to form $H^\ell_{n,k}$ for $0 < \ell \leq \frac{n-2}{2}$ and if nk is odd, we can add the edges $\left\{\{i, i + \frac{n-2}{2}\} : 0 \leq i < \frac{n-1}{2}\right\}$ to form $H^\ell_{n,k}$ for $0 < \ell \leq \frac{n-1}{2}$. $\square$

With this notation, we create a system to realize $(\kappa, \delta, \delta, \Delta)$. For the remainder of the paper, every realization will be composed of three subgraphs H, L, and M with $|L| \leq |M|$, visualized as

$$L \equiv H \equiv M.$$

In this representation, H is the set of cut vertices, so it follows that $|H| = \kappa$. Visually we can see that $[L, M] = \emptyset$ and L and M are the remaining subgraphs after removing the cut vertices. For consistency we denote the vertices in L by $\{u_i\}_{i=0}^{|L|-1}$, the vertices in M by $\{v_i\}_{i=0}^{|M|-1}$, and the vertices in H by $\{s_i\}_{i=0}^{|H|-1}$.

3 Basic Results

In this section we prove facts about realizing our graphs. Given any 4-tuple we wish to answer if the parameters are realizable for any order n, create an algorithm to produce a graph of that order, and explain the relationships where our realizations can be made for small n. We will begin with parameters for which no realization exists.

Theorem 3.1. *There is no possible realization for* $(2, \delta, \delta, \delta)$ *where* δ *is odd.*

Proof. Given a graph G with $\kappa = 2$ and $\lambda = \delta = \Delta$ all odd, let $H = \{s_1, s_2\}$ and $|[s_1, L]| = a$, $|[s_1, M]| = b$, $|[s_2, L]| = c$, and $|[s_2, M]| = d$. Since $s_1 s_2$ may be an edges and $\rho(s_1) = \rho(s_2) = \delta$,

$$a + b \leq \delta,$$

$$c + d \leq \delta.$$

But δ is odd so either, $a \leq \frac{\delta-1}{2}$ or $b \leq \frac{\delta-1}{2}$. Likewise, $c \leq \frac{\delta-1}{2}$ or $d \leq \frac{\delta-1}{2}$. Thus, we can find $x \in \{a, b\}$ and $y \in \{c, d\}$ where $x \leq \frac{\delta-1}{2}$ and $y \leq \frac{\delta-1}{2}$. Then $x + y \leq \delta - 1$, but with the corresponding vertices removed G is a disconnected graph. Therefore, these parameters are not realizable. $\square$

Given any other $(\kappa, \delta, \delta, \Delta)$ we can realize a graph for any $n \geq 2\delta + 2$. An exception to this assertion is when the degree of our graph is odd for all vertices, where $\delta = \Delta$ and δ is odd. Because of the Handshaking Lemma, we can realize our graph only for even $n \geq 2\delta + 2$.

Theorem 3.2. *Given $\kappa, \delta, \delta, \Delta$ with $\kappa + \Delta < 2\delta$, our realization function that determines the possible range of n for realizing our parameters is*

$$F(\kappa,\delta,\delta,\Delta) = \begin{cases} n \geq 2\delta+2 & \text{if} \quad \Delta < \frac{\delta}{\kappa}+1+\delta-\kappa, \\ n \geq 2\delta+2-\kappa+m & \text{if} \quad \frac{\delta+\kappa+\kappa\delta+\kappa^2}{\kappa} \leq \Delta < 2(\delta+1-\kappa), \\ n \geq 2\delta+2-\kappa & \text{if} \quad 2(\delta+1-\kappa) \leq \Delta, \\ \emptyset & \text{if} \quad (2,\delta,\delta,\delta), \text{where } \delta\text{-odd}, \end{cases}$$

where m is the ceiling of the solution to the quadratic equation $f(x) = \kappa(\Delta + \kappa - \delta - 1) - (\delta+1)x + x^2$.

Our paper will be organized as follows. In Section 4 and Section 5, we will describe how to realize graphs with order $n \geq 2\delta + 2$. This will be the basis of our paper since we will show that any $(\kappa, \delta, \delta, \Delta)$ is realizable in this range. After the main analysis, in Sections 6 and 7 we will elucidate situations where there are possible realizations smaller than $n = 2\delta + 2$. In Section 5, We will prove that for any given 4-tuple, the smallest possible realization of a graph is $n = 2\delta+2-\kappa$. Our algorithms that realize the minimum values of n will examine the range from $n = 2\delta + 2 - \kappa$ to $n = 2\delta + 2$. In section 6, for any $(\kappa, \delta, \delta, \Delta)$ we will determine the possible minimum realization, and then will have a complete algorithm for any size, n, greater than or equal to the smallest realization. With multiple cases we will partition the entire set of 4-tuples $(\kappa, \delta, \delta, \Delta)$.

4 Realizing Parameters of Size $n \geq 2\delta + 2$

In this section we realize any $(\kappa, \delta, \delta, \Delta)$ for size $n = 2\delta+2+c$ with $c \geq 0$, where c is even if $\delta = \Delta$ and δ is odd. For any given parameters we realize a graph as

$$L \equiv H \equiv M,$$

with $L = K_{\delta+2-\kappa}$, $H = H^r_{\kappa,q}$, and $M = H^{\beta}_{\delta+c,\delta-1}$ where $q < \kappa$ and $0 \leq r < \frac{\kappa}{2}$. Note that for any parameters, L remains a fixed order while M is varied based on c. To define q, r, and β we consider the two cases $\delta = \Delta$ and $\delta < \Delta$.

Given these cases, we will look at the relationship of $\delta + c$ to determine a possible realization by defining the unknown variables in our algorithm. Note that $\delta + c$ is the number of vertices in M.

Edge Connectivity

We will first show that for all of our realizations the edge connectivity is δ. For our realization

$$K_{\delta+2-\kappa} \equiv H^r_{\kappa,q} \equiv H^{\beta}_{\delta+c,\delta-1},$$

we attach each $u_i \in L$ to $\kappa - 1$ vertices in H, to force $\rho(u_i) = \delta$. Then the number of edges connecting L to H is $(\delta + 2 - \kappa)(\kappa - 1)$. Since for $\kappa > 2$,

$$(\delta + 2 - \kappa)(\kappa - 1) - \delta = \delta(\kappa - 2) + (\kappa - 1)(\kappa - 2)),$$

it follows that $(\delta + 2 - \kappa)(\kappa - 1) \geq \delta$.

This proves that there are sufficient edges connecting L and H. We will now consider the connections between H and M. The minimum number of edges that can be connected to H is $\kappa\delta$, so we must show that the cut vertices have enough space to attach at least δ vertices from M after attaching L. The remaining number of connecting edges is at least

$$\kappa\delta - (\kappa - 1)(\delta + 2 - \kappa) = (\kappa - 1)(\kappa - 2) + \delta \geq \delta.$$

Thus we have enough vertices in H to guarantee that we can successfully attach L and M while satisfying our minimum edge connectivity.

The Regular Case: $\delta = \Delta$

In this regular case, we consider the relationship of $\delta + c$ and $(\kappa - 1)(\kappa - 2) + \delta$.

4.1 $c < (\kappa - 1)(\kappa - 2)$

When the number of vertices in M is less than $(\kappa - 1)(\kappa - 2) + \delta$, we let $\beta = 0$ and our realization is

$$K_{\delta+2-\kappa} \equiv H^r_{\kappa,q} \equiv H_{\delta+c,\delta-1},$$

where

$$(\kappa - 1)(\kappa - 2) - c + r_{c(\delta+1)} = \kappa q + 2r$$

with $0 \leq 2r < \kappa$.

We obtain this equality by determining the edge set connecting L and M. Since the graph is regular, the total number of degrees in our cut vertices must be $\kappa\delta$. When we connect L and M to the cut vertices each $u_i \in L$ is connected to $\kappa - 1$ vertices in H and each $v_i \in M$ is connected to 1 vertex in H, which gives us $\rho(u_i) = \rho(v_i) = \delta$. In the case where δ is even and c is odd, then we do not attach v_0 to H since M is an irregular Harary graph with $\rho(v_0) = \delta$. These edges are attached to H evenly. Since we have $\delta + c < \kappa(\kappa - 3) + \delta + 2$, the edges connecting L and M to H is less than $\kappa\delta$ since

$$\kappa\delta - (\delta + c) - (\kappa - 1)(\delta + 2 - \delta) \geq \kappa\delta - \kappa(\kappa - 3) + \delta + 2 - (\kappa - 1)(\delta + 2 - \delta) \geq 0.$$

Thus the remaining edges in H needed to give our graph regular degree are given by q and r. Since

$$\kappa\delta - (\delta + c) - (\kappa - 1)(\delta + 2 - \delta) + r_{c(\delta+1)} = \kappa(\kappa - 3) + 2 - c + r_{c(\delta+1)},$$

our equation $\kappa(\kappa - 3) + 2 - c + r_{c(\delta+1)} = \kappa q + 2r$ gives us the conditions for the remaining edges of the cut vertices. Note that $\kappa(\kappa - 3) + 2 - c + r_{c(\delta+1)}$ is always even, for if c is odd, then δ must be even. So there will always exist a q and r to satisfy the remainder, where $\kappa > q$.

4.2 $c \geq (\kappa - 1)(\kappa - 2)$

For this case the number of vertices in M are sufficiently large so we let $q = r = 0$ and realize our parameters as

$$K_{\delta+2-\kappa} \equiv H_{\kappa,0} \equiv H^{\beta}_{\delta+c,\delta-1},$$

where

$$\beta = \frac{c - (\kappa - 1)(\kappa - 2) - r_{c(\delta+1)}}{2}.$$

The utility of β is to determine the number of remaining vertices in M that are not connected to H. Similarly to the $\delta + c < \kappa(\kappa - 3) + \delta + 2$ case, each $u_i \in L$ is connected to $\kappa - 1$ cut vertices defined as $[u_i, H^u_i]$ where

$$H^u_i = \{s_{\kappa-i}, \dots, s_{2\kappa-i-2}\}.$$

We connect each $v_j \in M$, where $j \in \{1, \dots, (\kappa - 1)(\kappa - 2)\}$ to a single $s_k \in H$ where

$$k = (\delta + 2 - \kappa)(\kappa - 1) + j - 1.$$

The remaining edges in M of degree $\delta - 1$ we connect together with the edges counted by β. We are assured that β is always a positive whole number since $c - (\kappa - 1)(\kappa - 2) - r_{c(\delta+1)}$ is always even. When δ is odd, our Harary graph is regular with an even number of vertices to connect. If δ is even, we note that when c is odd we have an irregular Harary graph with $\rho(v_0) = \delta$. Note that $r_{c(\delta+1)}$ forces $c - (\kappa - 1)(\kappa - 2) - r_{c(\delta+1)}$ to be even and we connect the remaining vertices.

4.3 Example: $(4, 6, 6, 6)$ for $n = 19$

We first note that the parameters $(4, 6, 6, 6)$ can be realize for size $n = 2\delta + 2 + c = 14 + c$. For $n = 19$, $c = 5$,

$$5 = c < (\kappa - 1)(\kappa - 2) = 6,$$

so from (4.1), we realize $(4, 6, 6, 6)$ with $n = 19$ as

$$K_4 \equiv H^+_{4,0} \equiv H_{11,5}.$$

This represents the graph in Figure 1.

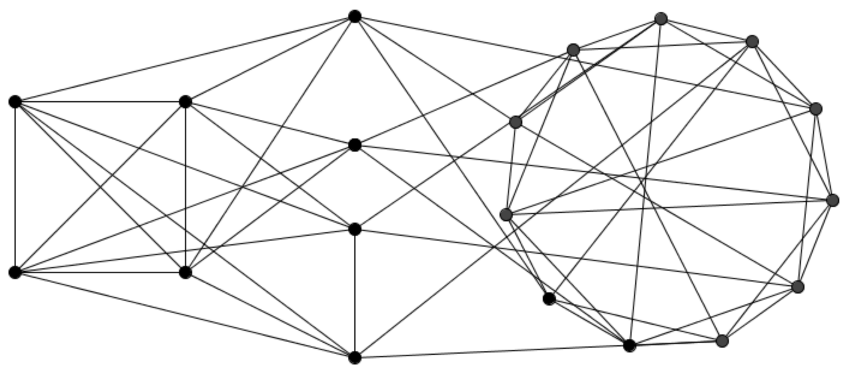

Figure 1

5 Realizing parameters for $n \geq 2\delta+2$ given $\delta < \Delta$

For realizing $n = 2\delta+2+c$, we recall that $|L| = \delta+2-\kappa$ and $|M| = \delta+c$. When $\delta < \Delta$, we have two cases that determine the number of edges in H and $[H, M]$. In every case we evenly attach each $u_i \in L$ to $\kappa - 1$ cut vertices. Subsequently

$$|[L, H]| = (\delta + 2 - \kappa)(\kappa - 1) = \delta(\kappa - 1) - (\kappa - 1)(\kappa - 2)$$

is constant with respect to the order of the graph. Therefore, while presenting these cases we will describe the set of connecting edges $[H, M]$, any q, r edges that we may add to H, and the addition of β edges to M if necessary. The first case examines small $\delta + c$ which breaks into two subcases. The second case is for large $\delta + c$, where in our analysis we must add edges to M for some vertices not adjacent to H.

5.1 Case 1: $\frac{\kappa(\Delta-\delta)+\delta+(\kappa-1)(\kappa-2)}{\delta+c} > 1$

Given $\frac{\kappa(\Delta-\delta)+\delta+(\kappa-1)(\kappa-2)}{\delta+c} > 1$, there are edges that must be added to H or $[H, M]$ to guarantee minimum and maximum degree. In this case $\beta = 0$. We examine this case by analyzing the relationship between $\frac{\Delta+1(\kappa-1)(\kappa-2)}{\delta+c}$ and $\Delta - \delta + 1$.

5.1.1 $\frac{\Delta+(\kappa-1)(\kappa-2)}{\delta+c} > \Delta - \delta + 1$

In the first subcase the nontrivial variables are q and r. We attach each $u_j \in L$ to $\kappa - 1$ cut vertices so that each $\rho(u_i)$ has degree δ. We realize our parameters as

$$K_{\delta+2-\kappa} \equiv H_{\kappa,q} \equiv H_{\delta+c,\delta-1}.$$

In this case we will explain how to add edges to H and $[H, M]$. This subcase gives two possibilities.

(i) $\Delta - \delta + 1 < \kappa$

In this case our quotient is different than the other cases we have experienced. We realize our parameters as

$$K_{\delta+2-\kappa} \equiv H_{\kappa,\delta-q} \equiv H_{\delta+c,\delta-1},$$

where $0 \leq q$ and $0 \leq 2r < \kappa$ are given by

$$(\kappa-1)(\delta+2-\kappa) + (\Delta+1-\delta)(\delta+c) - r_{c(\delta+1)} = \kappa q + r.$$

We wish to force each $u_i \in L$ to have degree δ and each $v_i \in M$ to have degree Δ. To do this, each u_i is connected to $\kappa - 1$ cut vertices and each v_i is connected to $\Delta + 1 - \delta$ cut vertices with modular arithmetic. We have $(\kappa-1)(\delta+2-\kappa) + (\Delta+1-\delta)(\delta+c) - r_{c(\delta+1)}$ edges connecting L and M to our cut vertices. Our quotient used to define the degree of H determines the number of edges to which each vertex in H is attached from L and M, and $\delta - q$ denotes the remainder of edges to ensure minimum degree. Therefore, there will be $\kappa - r$ vertices in H with degree δ and r vertices of degree $\delta + 1$.

(ii) $\Delta - \delta + 1 \geq \kappa$

In this case we realize our parameters as

$$K_{\delta+2-\kappa} \equiv H_{\kappa,q} \equiv H_{\delta+c,\delta-1},$$

where $0 \leq q$ and $0 \leq 2r < \kappa$ are given by

$$\Delta + (\kappa-1)(\kappa-2) - \kappa(\delta+c) = \kappa q + r.$$

In this case we attach each $v_i \in M$ to each cut vertex. The quotient defines the remaining edges needed in H to give one vertex degree Δ and the others minimum degree. We can show that after attaching L and M to H, our q is less than κ. After connecting M, each v_i has degree $\delta - 1 + \kappa$ which is less than Δ.

5.1.2 $\frac{\Delta+(\kappa-1)(\kappa-2)}{\delta+c} \leq \Delta - \delta + 1$

For this case, let $q = r = \beta = 0$. Our realization is

$$K_{\delta+2-\kappa} \equiv H_{\kappa,0} \equiv H_{\delta+c,\delta-1}.$$

First note that in this case we have $\delta+c \leq [H,M] \leq (\delta+c)\cdot\min\{\Delta-\delta+1, \kappa\}$. To realize our parameters we manipulate the size of $[H,M]$ so that for any $s_k \in H$, $\rho(s_0) = \Delta$, $\delta \leq \rho(s_k) \leq \Delta$ for $k \neq 0$, and $\delta \leq \rho(v_j) \leq \Delta$ for any $v_j \in M$. We connect M to H until our cut vertices have the desired degree, and it follows that for any $v_j, v_k \in M$, both $\delta \leq \rho(v_j) \leq \rho(v_k) \leq \Delta$ and $|v_j - v_k| \leq 1$. We must analyze two subcases here. In both cases we will show that the edges $[H,M]$ can be large enough to satisfy maximum and minimum degree in both H and M.

Case 1: $\frac{\Delta+(\kappa-1)(\kappa-2)}{\delta+c} < 1$

In the special situation where $\frac{\Delta+(\kappa-1)(\kappa-2)}{\delta+c} < 1$, each $v_j \in M$ is connected to a single vertex so that $[H,M] = \delta + c$. These $\delta + c$ edges are connected evenly over H such that for any $s_k \in H$, $\rho(s_0) = \Delta$ and $\rho(s_k) = \delta$ where $k \neq 0$. There are sufficient edges given $\Delta + (\kappa-1)\delta - [L,H] = \Delta + (\kappa-1)(\kappa-2) < \delta + c$. The remaining edges are attached evenly over s_k where $k \neq 0$. These cut vertices will not exceed the maximum degree since

$$\kappa\Delta - [L,H] - (\delta + c) = \kappa(\Delta - \delta) + \delta + (\kappa - 1)(\kappa - 2) - (\delta + c)$$

and in Case 1, $\frac{\kappa(\Delta-\delta)+\delta+(\kappa-1)(\kappa-2)}{\delta+c} > 1$. Thus the connecting edges between H and M give each vertex in M degree δ, and for any $s_k \in H$, $\rho(s_0) = \Delta$ and $\delta \leq \rho(s_k) \leq \Delta$ for $k \neq 0$.

Case 2: $\frac{\Delta+(\kappa-1)(\kappa-2)}{\delta+c} \geq 1$

In the general case we have $\frac{\Delta+(\kappa-1)(\kappa-2)}{\delta+c} \geq 1$. We must attach M to H in the following manner. For $s_k \in H$, we wish to make $\rho(s_0) = \Delta$ and $\rho(s_k) = \delta$ where $k \neq 0$. After attaching L to H, the edges needed to attach to H to guarantee these degrees is

$$\begin{aligned}
&\Delta + (\kappa - 1)\delta - (\delta + 2 - \kappa)(\kappa - 1) \\
&\quad = \Delta + (\kappa - 1)\delta - \kappa\delta + \delta + \kappa^2 - 3\kappa + 2 \\
&\quad = \Delta + (\kappa - 1)(\kappa - 2).
\end{aligned}$$

Since we are assuming $\frac{\Delta+(\kappa-1)(\kappa-2)}{\delta+c} \geq 1$, there is room in H such that we can attach each $v_j \in M$ to H giving v_j degree δ. We will continue connecting H to M evenly over both H and M until $[H,M] = \delta + (\kappa - 1)(\kappa - 2)$, and after each cut vertex has degree δ, attach $\Delta - \delta$ more edges to s_0 to give $[H, M] = \Delta + (\kappa - 1)(\kappa - 2)$ and $\rho(s_0) = \Delta$. We will show that there are sufficient edges for the cut vertices to have the desired degree.

Suppose that $\Delta - \delta + 1 < \kappa$. Hence it is possible to attach each $v_j \in M$ to at most $\Delta - \delta + 1$ cut vertices in H. This proves to be sufficient since by our inequality:

$$\frac{\Delta + (\kappa-1)(\kappa-2)}{\delta + c} \leq \min\{\Delta - \delta + 1, \kappa\}$$
$$\frac{\Delta + (\kappa-1)(\kappa-2)}{\delta + c} \leq \Delta - \delta + 1$$
$$\Delta + (\kappa-1)(\kappa-2) \leq (\delta + c)(\Delta - \delta + 1).$$

Suppose that $\kappa < \Delta - \delta + 1$. Now, when we attach a single $v_j \in M$ to the cut vertices, because we can maximally attach v_j to κ vertices, $\rho(v_j) < \Delta$. With the same steps we show

$$\frac{\Delta + (\kappa-1)(\kappa-2)}{\delta + c} \leq \min\{\Delta - \delta + 1, \kappa\}$$
$$\frac{\Delta + (\kappa-1)(\kappa-2)}{\delta + c} \leq \kappa$$
$$\Delta + (\kappa-1)(\kappa-2) \leq (\delta + c)\kappa.$$

Therefore, for whichever value $\min\{\Delta - \delta + 1, \kappa\}$ takes, we can connect M to H in a way that gives $\rho(s_0) = \Delta$ and $\rho(s_k) = \delta$ for $k \neq 0$. We also are assured that $\delta \leq \rho(v_j) \leq \Delta$ for any $v_j \in M$.

5.2 Case 2: $1 \geq \frac{\kappa(\Delta-\delta)+\delta+(\kappa-1)(\kappa-2)}{\delta+c}$

In this case, c is large enough so that after evenly attaching each vertex in M to H, $[H, M] \geq \kappa\Delta - [L, H]$. This implies that some cut vertex would exceed the maximum degree, so every vertex in M cannot be attached to H. To begin, we set $q = r = 0$. We attach M to H so that each cut vertex has degree Δ, which gives us that

$$[H, M] = \kappa\Delta - [L, H] = \kappa(\Delta - \delta) + \delta + (\kappa - 1)(\kappa - 2).$$

We attach all edges and then must account for the extra $v_i \in M$ that exceed this amount. Each $v_j \in M$ with $j \in \{0, \ldots, \kappa(\Delta-\delta)+\delta+(\kappa-1)(\kappa-2)-1\}$ is attached to H to give each cut vertex degree Δ. The remaining edges are attached by the β edges added to M. Since there are $\delta + c - \kappa(\Delta - \delta) - \delta - (\kappa - 1)(\kappa - 2)$ vertices in M with degree $\delta - 1$, the number of edges we must add is

$$\beta = \left\lceil \frac{\delta + c - \kappa(\Delta - \delta) - \delta - (\kappa - 1)(\kappa - 2)}{2} \right\rceil.$$

After adding β edges to M, each $v_j \in M$ has degree δ or $\delta + 1$.

6 Minimum Realizations of $(\kappa, \delta, \delta, \Delta)$

The first theorem we need is well-known but we also give a proof.

Theorem 6.1. *Given any $(\kappa, \delta, \delta, \Delta)$, we cannot realize a graph of order $n < 2\delta + 2 - \kappa$.*

Proof. If we have a graph of order $n < 2\delta + 2 - \kappa$, then $|L| \leq \delta - \kappa$, since $|H| = \kappa$ and $|L| + |M| \leq 2(\delta - \kappa) + 1$. This means that each $u_i \in L$ has at most degree $\delta - \kappa - 1$. Since L is only adjacent to H, it can only connect to the set of cut vertices H. Hence we cannot satisfy the minimum degree δ for the vertices in L, since there are κ cut vertices. Therefore, any adjacent graph to the cut vertices must have at least order $\delta + 1 - \kappa$. $\square$

To determine the smallest possible realization for $(\kappa, \delta, \delta, \Delta)$, we will be analyzing the relationship between Δ and $f(\kappa, \delta)$, where f will be a function of κ and δ. There are three cases. We note that our possible realizations become smaller for relatively small $\delta - \kappa$ and relatively large $\Delta - \delta$. In the first case, we can realize a minimum $n = 2\delta + 2 - \kappa$. In the second case, there is a quadratic equation that presents the minimum m for which we can realize our graph of size $n = 2\delta + 2 - \kappa + m$, for $m \in \{1, \ldots, \kappa - 1\}$. The final case is $(\kappa, \delta, \delta, \Delta)$ with no realization smaller than $n = 2\delta + 2$. Given the minimum realization as $n = 2\delta + 2 - \kappa + m$, we then can find the $c \in \{0, \ldots, \kappa - 1 - m\}$ that we can realize the parameters for size $n \in \{2\delta + 2 - \kappa + m, \ldots, 2\delta + 1\}$.

6.1 $\Delta \geq 2(\delta + 1 - \kappa)$: Realizing $n = 2\delta + 2 - \kappa$

In this case we can realize $(\kappa, \delta, \delta, \Delta)$ for $n = 2\delta + 2 - \kappa$. Given $\Delta \geq (2\delta + 1 - \kappa)$, we will consider realizations in the interval $2\delta + 2 - \kappa \leq n < 2\delta + 2$. To analyze the minimum case we will consider $\delta = \Delta$ and $\delta < \Delta$.

6.1.1 $\delta = \Delta$

We first note that when $\delta = \Delta$, our inequality to denote possible minimum realizations becomes

$$2\kappa - 2 \geq \delta.$$

Since the degree is regular in our realizations, if δ is odd, we only consider even n. Furthermore, if $\kappa\delta$ is odd, then $2\delta + 2 - \kappa$ is odd, which is not realizable. We will now show how to realize the range of minimum n for parameters that satisfy our inequality.

Theorem 6.2. *Given $(\kappa, \delta, \delta, \delta)$ that satisfy*

$$2\kappa - 2 \geq \delta,$$

and given some $c \in \{0, ..., \kappa - 1 - r_{\kappa\delta}\}$, we can realize our parameters of size $n = 2\delta + 2 - \kappa + c$ where c is the same parity as $\kappa\delta$, as

$$K_{\delta+1-\kappa} \equiv H^r_{\kappa,q} \equiv K_{\delta+1-\kappa+c+r_{\kappa\delta}}$$

where $0 \leq 2r < \kappa$ such that

$$\kappa(\kappa - 1) - (\delta + 1 - \kappa + c)(\kappa - c) = \kappa q + 2r.$$

We attach L to H by connecting each $u_i \in L$ to each cut vertex in H and connect each $v_j \in M$ to $\kappa - c$ distinct vertices in H, such that after connecting M to H, for any two $s_\ell, s_k \in H$, $|\rho(s_\ell) - \rho(s_k)| \leq 1$. Our equation to determine q and r is the quotient of edges remaining in H to guarantee that for all $s_k \in H$, $\rho(s_k) = \delta$. To derive the equation for q and r, we note that we need $\kappa\delta$ edges in H to satisfy regular degree. After connecting L and M we see that

$$\kappa\delta - (\delta + 1 - \kappa)\kappa - (\delta + 1 - \kappa + c)(\kappa - c)$$
$$= \kappa(\kappa - 1) - (\delta + 1 - \kappa + c)(\kappa - c).$$

Thus, this is the number of edges we must fill in H to force the degree to be regular. Our equation

$$\kappa(\kappa - 1) - (\delta + 1 - \kappa + c+)(\kappa - c) = \kappa q + 2r$$

designates these remaining edges. We note that the number of edges remaining is always even for whatever the parity of κ and δ. We can also show that $\kappa(\kappa - 1) - (\delta + 1 - \kappa + c)(\kappa - c)$ is always positive. Let $f(c)$ denote the number of edges attached from M to H. If

$$f(c) = (\delta + 1 - \kappa + c)(\kappa - c),$$

then differentiating with respect to c gives

$$f'(c) = 2\kappa - \delta - 1 - 2c.$$

Hence the critical number occurs when

$$c = \frac{\delta + 1 - 2\kappa}{2}.$$

Since $2\kappa - 2 \geq \Delta$ in this case, this value is at most $\frac{1}{2}$. So the most edges must be attached when $|M| = \delta + 1 - \kappa$. We can show that this is less than $\kappa(\kappa - 1)$;

$$\begin{aligned}
\kappa(\kappa - 1) &\geq (\delta + 1 - \kappa)\kappa \\
\kappa(\kappa - 1) &\geq \kappa\delta - \kappa(\kappa - 1) \\
2\kappa(\kappa - 1) &\geq \kappa\delta \\
2\kappa - 2 &\geq \delta.
\end{aligned}$$

Thus, our q will always be positive.

6.1.2 $\delta < \Delta$

For $\delta < \Delta$ with $\Delta \geq 2\delta + 1 - \kappa$, we can realize our parameters in the range $n = 2\delta + 2 - \kappa + c$ for $c \in \{0, \ldots, \kappa - 1\}$. Unlike the regular case, our realization changes given the size of c.

Theorem 6.3. *Given $(\kappa, \delta, \delta, \Delta)$ that satisfy*

$$\Delta \geq (2\delta + 1 - \kappa),$$

and given some $c \in \{0, \ldots, \kappa - 1\}$, we can realize our parameters of size $n = 2\delta + 2 - \kappa + c$ in two cases.

Case 1: *If*

$$\kappa(2\delta + 1 - \kappa - \Delta) \geq c(\delta + 1 - 2\kappa + c),$$

then we realize our parameters as

$$K_{\delta+1-\kappa} \equiv H_{\kappa,q}^{r-r_{\kappa q}} \equiv K_{\delta+1-\kappa+c},$$

where

$$\kappa\delta - (\delta + 1 - \kappa)\kappa - (\delta + 1 - \kappa + c)(\kappa - c) = \kappa q + 2r + r_{c\delta(1+\kappa)}.$$

Case 2: *If*

$$\kappa(2\delta + 1 - \kappa - \Delta) \geq c(\delta + 1 - 2\kappa + c),$$

then we realize our parameters as

$$K_{\delta+1-\kappa} \equiv K_\kappa \equiv K_{\delta+1-\kappa+c}.$$

The second case is sometimes not applicable for any c. Its utility comes when c becomes large enough and there are so few edges that q becomes larger than κ. We will begin by analyzing the first case.

Case 1: $\kappa(2\delta + 1 - \kappa - \Delta) \geq c(\delta + 1 - 2\kappa + c)$

Given the parameters and c that satisfy this equation, we connect L and M to H to guarantee that all vertices in L and M have degree δ. To attach each $u_i \in L$, we attach u_i to every cut vertex in H guaranteeing that $\rho(u_i) = \delta$. We then connect each $v_j \in M$ to $\kappa - c$ distinct vertices in H, such that after connecting M to H, for any two $s_\ell, s_k \in H$, $|\rho(s_\ell) - \rho(s_k)| \leq 1$. Thus all vertices in L and M have degree δ.

After connecting L and M, q and r are used to force each $s_k \in H$ to have degree Δ. We will show that $q \leq \kappa - 1$. We define q and r by

$$\kappa\delta - (\delta + 1 - \kappa)\kappa - (\delta + 1 - \kappa + c)(\kappa - c) = \kappa q + 2r + r_{c\delta(1+\kappa)}.$$

We show that the remaining edges needed to guarantee maximum degree in H after connecting L and M (the left side of the equation) is less than $\kappa(\kappa - 1)$ and so it follows that $q \leq \kappa - 1$. Therefore

$$\begin{gathered}
\kappa(\kappa - 1) \geq \kappa\delta - (\delta + 1 - \kappa)\kappa - (\delta + 1 - \kappa + c)(\kappa - c), \\
\kappa\delta - \kappa\Delta \geq -(\delta + 1 - \kappa + c)(\kappa - c), \\
\kappa\delta - \kappa\Delta \geq (\delta + 1 - \kappa + c)(c - \kappa), \\
\kappa\delta - \kappa\Delta \geq c(\delta + 1 - \kappa + c) - \kappa(\delta + 1 - \kappa) - c\kappa, \\
\kappa\delta - \kappa\Delta + \kappa(\delta + 1 - \kappa + c) \geq c(\delta + 1 - \kappa + c) - c\kappa, \\
\kappa(2\delta + 1 - \kappa - \Delta) \geq c(\delta + 1 - 2\kappa + c).
\end{gathered}$$

Thus, our q will exist to satisfy the equation. Also, note that if H is a Harary graph with κq odd, then $r_{\kappa q}$ takes one degree away from the vertex of degree $\Delta +$ 1 and another vertex so our maximum degree holds. The number of remaining edges $\kappa\delta - (\delta + 1 - \kappa)\kappa - (\delta + 1 - \kappa + c)(\kappa - c)$ is odd only if c and δ are odd and κ is even, so $r_{c\delta(1+\kappa)}$ guarantees that that the equation is satisfied.

Case 2: $\kappa(2\delta + 1 - \kappa - \Delta) < c(\delta + 1 - 2\kappa + c)$

For this case we must keep our edges in H constant. We attach L to H in a similar manner, so each u_i connects to all of H so that each u_i is given degree δ. We then note that each $s_k \in H$ has degree $\kappa - 1 + \delta + 1 - \kappa = \delta$. We wish to give each s_k degree Δ. So we need $\kappa(\Delta - \delta)$ edges from M to attach to H. We first find $0 \leq a_r < \delta + 1 - \kappa + c$ such that

$$a_r \equiv \kappa(\Delta - \delta)(\mod \delta + 1 - \kappa + c),$$

and then find a_q such that

$$\kappa(\Delta - \delta) = (\delta + 1 - \kappa + c)a_q + a_r.$$

We partition M into two sets, $M_{a_q+1} = \{v_0, \dots, v_{a_r-1}\}$ and $M_{a_q+1} = \{v_{a_r}, \dots, v_{\delta-\kappa+c}\}$ such that the $v_i \in M_{a_q+1}$ are attached to $a_q + 1$ vertices in H in a modular fashion and $v_i \in M_{a_q}$ are attached to a_q vertices in the same way. Thus each $s_k \in H$ has degree Δ. We will now prove two things. First, we will show that in H, there are enough vertices to attach to each $v_j \in M$ so we can satisfy the minimum degree. Second, we will show that there are enough edges in M to ensure that each cut vertex will have degree Δ.

Proof. **Minimum Degree in M**
We first note that there are a maximum of $\kappa(\Delta - \delta)$ edges in $[M, H]$ bounded by our maximum degree, since our maximum is

$$\sum_{s_i \in H} (\Delta - \rho(s_i)) = \kappa(\Delta - (\kappa - 1 + \delta + 1 - \kappa)) = \kappa(\Delta - \delta).$$

To prove our claim, we must show that $\kappa(\Delta - \delta)$ is greater than the number of edges needed to satisfy the minimum degree in M. We can show that

$$\kappa(\Delta - \delta) \geq (\delta + 1 - \kappa + c)(\kappa - c)$$

$$\kappa(\Delta - \delta) \geq \kappa(\delta + 1 - \kappa) + c\kappa - (\delta + 1 - \kappa + c)c$$

$$\kappa(\Delta + \kappa - 2\delta - 1) \geq (2\kappa - \delta - 1 - c)c$$

and finally multiplying through by -1 we see that

$$\kappa(2\delta + 1 - \kappa - \Delta) < c(\delta + 1 - 2\kappa + c).$$

Therefore, each $u_j \in M$ has at least degree δ. $\square$

We will now show that there are enough edges in $[M, H]$ so that each $s_i \in H$ has degree Δ.

Proof. **Maximum Degree in H**
We have shown in the previous proof that we need $\kappa(\Delta - \delta)$ edges to guarantee that each $s_k \in H$ has degree Δ. We now note that each vertex in M can be attached to a maximum of $\Delta - \delta + \kappa - c$ vertices in κ. So we can show that

$$\kappa(\Delta - \delta) \leq (\delta + 1 - \kappa + c)(\Delta + \kappa - \delta - c),$$

$$c(\delta + 1 - \kappa + c) - c(\Delta + \kappa - \delta) \leq (\delta + 1 - \kappa)(\Delta + \kappa - \delta) - \kappa(\Delta - \delta),$$

$$c(2(\delta + \frac{1}{2} - \kappa) - \Delta)) \leq (\delta + 1 - \kappa)(\Delta + \kappa - \delta) - \kappa(\Delta - \delta).$$

Since $\Delta \geq 2(\delta + 1 - \kappa)$, we have

$$0 \leq (\delta + 1 - \kappa)(\Delta + \kappa - \delta) - \kappa(\Delta - \delta).$$

Hence we must show that the right-hand-side of the above is positive. To that end, note that

$$\begin{aligned}
&(\delta+1-\kappa)(\Delta+\kappa-\delta)-\kappa(\Delta-\delta)\\
&=\delta\Delta+\Delta-2\kappa\Delta+3\kappa\delta-\delta-\delta^2+\kappa-\kappa^2\\
&=(\delta+1)(\Delta)-2\kappa\Delta+3\kappa\delta-\delta(\delta+1)-\kappa(\kappa-1)\\
&=(\delta+1)(\Delta-\delta)-2\kappa\Delta+3\kappa\delta-\kappa(\kappa-1).
\end{aligned}$$

Since $2\delta > \kappa + \Delta$ it follows that

$$\begin{aligned}
&(\delta+1)(\Delta-\delta)-2\kappa\Delta+3\kappa\delta-\kappa(\kappa-1)\\
&<(\delta+1)(\Delta-\delta)-2\kappa\Delta+\kappa\delta(\kappa+\Delta)-\kappa(\kappa-1)\\
&=(\delta+1)(\Delta-\delta)+\kappa(\delta(\kappa+\Delta)-2\Delta-\kappa+1),
\end{aligned}$$

which is clearly positive. Thus, we have proven that there are enough edges. □

Through these proofs, we have shown that after attaching M to H the degree of the vertices in M are in the range of δ and Δ, and before that we spelled out what the exact degrees were.

7 Parameters with minimum realizations of order $n > 2\kappa+2-\kappa$: $2(\delta+1-\kappa) > \Delta \geq \frac{\delta}{\kappa}+1+\delta-\kappa$

In this section we will be looking at $(\kappa, \delta, \delta, \Delta)$ that satisfy $2(\delta+1-\kappa) > \Delta \geq \frac{\delta}{\kappa}+1+\delta-\kappa$. In this case there exists some $m \in \{1, \ldots, \kappa\}$ for which we can realize $(\kappa, \delta, \delta, \Delta)$ of minimum order $n = 2\delta + 2 - \kappa + m$. We will begin by showing that given parameters that satisfy $2(\delta + 1 - \kappa) > \Delta$, then there is no realization of order $2\delta + 2 - \kappa$.

Theorem 7.1. ***If*** $2(\delta+1-\kappa) > \Delta$***, then no realization exists of size*** $n = 2\delta + 2 - \kappa$***.***

Proof. By the proof of Theorem 6.1, we know that $\delta + 1 - \kappa \leq |L| \leq |M|$. If we assume $n = 2\delta + 2 - \kappa$, we realize our parameters as

$$K_{\delta+1-\kappa} \equiv H^r_{\kappa,q} \equiv K_{\delta+1-\kappa},$$

for some $0 \leq q < \kappa$ and $0 \leq 2r < \frac{\kappa}{2}$. We note that the maximum number of edges we can connect to H is $\kappa\Delta$ and this occurs when $q = r = 0$. To satisfy the minimum degree in L and M, the connecting sets of edges have size $|[L, H]| + [M, H]| = \kappa(\delta+1-\kappa)+\kappa(\kappa+1-\delta) = 2\kappa(\delta+1-\kappa)$. This contradicts

$$\kappa\Delta < 2\kappa(\delta+1-\kappa)$$

$$\Delta < 2(\delta+1-\kappa).$$

Thus, given $2(\delta + 1 - \kappa) \geq \Delta$, we have $n > 2\delta + 2 - \kappa$. □

We will now show how we attain the minimum realization given $(\kappa, \delta, \delta, \Delta)$ with $2(\delta+1-\kappa) > \Delta$. We wish to find a and b that will realize our parameters as

$$K_a \equiv H_{k,q} \equiv K_b$$

minimally. To minimize n, we find that $|L| = a = \delta + 1 - \kappa$.

Theorem 7.2. ***For any*** $(\kappa, \delta, \delta, \Delta)$***, the smallest possible*** n ***is realized when*** $|L| = \delta + 1 - \kappa$***.***

Proof. Let $(\kappa, \delta, \delta, \Delta)$ be realized minimally by

$$K_a \equiv H^r_{k,q} \equiv K_b$$

where $a \leq b$ and $a + b + \kappa = n$. We wish to find a and b such that there are the fewest edges connected to H. The number of edges adjacent to L and M to satisfy the minimum degree is

$$(\delta - (a-1))a + (\delta - (b-1))b.$$

Let $\alpha = \delta + 1$ and $\beta = n - \kappa$. Then $b = \beta - a$, and we can rewrite the number of edges as a function g that is the number of edges adjacent to H in terms of a such that

$$g(a) = \beta(\alpha - \beta) + 2\beta a - 2a^2.$$

It follows that

$$g'(a) = 2\beta - 4a$$
$$g'(a) = 2n - 2\kappa - 4a$$
$$g'(a) = 2b - 2a \geq 0.$$

Thus, g is increasing for all a, so the smallest possible realization is for minimum a which means that $|L| = \delta + 1 - \kappa$. □

We have now shown that the smallest realization occurs when we realize our parameters as

$$K_{\delta+1-\kappa} \equiv H_{\kappa,q} \equiv K_b,$$

for some $b \in \{\delta + 2 - \kappa, \ldots, \delta\}$. The number of edges we connect to H from L and M so that any vertex in $L \cup M$ has minimum degree is

$$\kappa(\delta + 1 - \kappa) + (\delta - b + 1)b.$$

To minimize our realization we need to find the minimum b that after connecting L and M to H cannot exceed $\kappa\Delta$, where the cut vertices are all of maximum degree. We wish to find the minimum b that satisfies

$$0 \leq \kappa\Delta - (\delta + 1 - c)\kappa - b(\delta - b + 1).$$

This inequality gives us a quadratic equation that finds $b = |M|$. We define the function

$$M_{min}(x) = \kappa(\Delta + \kappa - \delta - 1) - (\delta + 1)x + x^2.$$

We note that $M'_{min}(x) = 2x - \delta - 1$, so $M_{min}(x)$ is increasing for all x since our range of values we consider are in the interval $[\delta + 2 - \kappa, \delta]$. We also note that

$$\begin{aligned}
M_{min}(\delta + 1 - \kappa) &= \kappa(\Delta + \kappa - \delta - 1) - (\delta + 1)(\delta + 1 - \kappa) + (\delta + 1 - \kappa)^2 \\
&= \kappa\Delta + 2\kappa^2 - \delta^2 - 2\kappa\delta - \kappa \\
&< 2\kappa(\delta + 1 - \kappa) + 2\kappa^2 - \delta^2 - 2\kappa\delta - \kappa = \kappa - \delta^2 < 0.
\end{aligned}$$

This follows from our proof, since if $|M| = \delta + 1 - \kappa$, then we cannot realize our parameters. We can also show that

$$\begin{aligned}
M_{min}(\delta) &= \kappa(\Delta + \kappa - \delta - 1) - (\delta + 1)\delta + \delta^2 = \kappa\Delta + \kappa^2 - \kappa\delta - \kappa - \delta \\
&< \kappa(2(\delta + 1 - \kappa)) + \kappa^2 - \kappa\delta - \kappa - \delta = \delta(\kappa - 1) - \kappa(\kappa - 1),
\end{aligned}$$

which is positive. Since $M_{min}(\delta + 1 - \kappa) < 0 < M_{min}(\delta)$, we know that on the interval there must exist some x^* such that $M_{min}(x^*) = 0$ by the Intermediate Value Theorem. We use this to determine the minimum $b = |M|$. We can determine the size of M as

$$b = \lceil x^* \rceil.$$

This is the smallest possible size of M that satisfies our inequality. At this point, we note that there exist $(\kappa, \delta, \delta, \delta)$ with odd δ that satisfy $2(\delta + 1 - \kappa) > \Delta \geq \frac{\delta}{\kappa} + 1 + \delta - \kappa$, but after solving the quadratic equation for the size of M, $b = \delta$. This gives $n = 2\delta + 1$ which is not realizable for odd regular degree. For such $(\kappa, \delta, \delta, \delta)$, there is no realization with $n < 2\delta + 2$. We will now state our theorem to find the minimum realization and to realize it from the minimum to $n = 2\delta + 1$.

Theorem 7.3. ***Minimum Realizations of*** $(\kappa, \delta, \delta, \Delta)$ ***given*** $2(\delta + 1 - \kappa) > \Delta \geq \frac{\delta}{\kappa} + 1 + \delta - \kappa$

Given $(\kappa, \delta, \delta, \Delta)$ *that satisfy*

$$2(\delta + 1 - \kappa) > \Delta \geq \frac{\delta}{\kappa} + 1 + \delta - \kappa,$$

then we can find the minimum realization of our parameters of size $n = 2\delta + 2 - \kappa + m$ *where* m *is defined as*

$$m = \left\lceil \frac{\delta + 1 + \sqrt{(\delta + 1)^2 - 4\kappa(\Delta + \kappa - \delta - 1)}}{2} \right\rceil - \delta + \kappa - 1.$$

We then realize $(\kappa, \delta, \delta, \Delta)$ *of order* $n = 2\delta + 2 - \kappa + m + c$*, where* $c \in \{0, \ldots, \kappa - m - 1\}$ *in two cases.*

Case 1: *If*
$$(\kappa - m - c)(\delta + 1 - \kappa + m + c) \geq \kappa(\Delta - \delta),$$
then we realize our parameters as
$$K_{\delta+1-\kappa} \equiv H_{\kappa,q}^{r-r_{\kappa q}} \equiv K_{\delta+1-\kappa+m+c},$$
where $q \leq \kappa - 1$ *and* $0 \leq 2r < \frac{\kappa}{2}$*, defined by*
$$\kappa\Delta - (\delta + 1 - \kappa)\kappa - (\delta + 1 - \kappa + m + c)(\kappa - m - c) = \kappa q + 2r + r^*$$
where
$$r^* = \begin{cases} 1 & \textit{if } \kappa \textit{ is even and } c \not\equiv m (\mod 2), \\ 0 & \textit{otherwise.} \end{cases}$$
Case 2: *If*
$$(\kappa - m - c)(\delta + 1 - \kappa + m + c) < \kappa(\Delta - \delta),$$
then we realize our parameters as
$$K_{\delta+1-\kappa} \equiv K_\kappa \equiv K_{\delta+1-\kappa+m+c},$$

The two cases are contingent on the number of edges needed in $[H, M]$ to give each vertex in M degree δ. We note that $|[L, H]|$ is constant and the number of edges is $\kappa(\delta + 1 - \kappa)$, so each $u_i \in L$ has degree δ.

In Case 1, we attach H and M with $q \leq \kappa - 1$ and $0 \leq 2r < \frac{\kappa}{2}$, for which we attach each $v_j \in M$ to $\kappa - m - c$ vertices in H which forces $\rho(v_j) = \delta$. In H, q and r give each $s_k \in H$ degree Δ. We can show that such q and r exist. After attaching L and M to H, to give each $s_k \in H$ degree Δ, there are $\kappa\Delta - (\delta + 1 - \kappa)\kappa - (\delta + 1 - \kappa + m + c)(\kappa - m - c)$ remaining degrees in H. We have already shown that m gives us the smallest order of L so that the remaining degrees needed is positive, and also that the number of edges from L that give each vertex in M degree δ, decreases with c. Since $q \leq \kappa - 1$, it suffices to show that $\kappa(\kappa - 1)$ is greater than the remaining degrees in H. To that end note

$$\kappa(\kappa - 1) \geq \kappa\Delta - (\delta + 1 - \kappa)\kappa - (\delta + 1 - \kappa + m + c)(\kappa - m - c),$$
$$(\delta + 1 - \kappa + m + c)(\kappa - m - c) \geq \kappa\Delta - (\delta + 1 - \kappa)\kappa - \kappa(\kappa - 1),$$
$$(\delta + 1 - \kappa + m + c)(\kappa - m - c) \geq \kappa(\Delta - \delta).$$

Thus, when c is small enough to satisfy this inequality, there will exist $q \leq \kappa - 1$ and $0 \leq 2r < \frac{\kappa}{2}$ that satisfy $\kappa\Delta - (\delta + 1 - \kappa)\kappa - (\delta + 1 - \kappa + m + c)(\kappa - m - c) = \kappa q + 2r + r^*$. Recall that

$$r^* = \begin{cases} 1 & \text{if } \kappa \text{ is even and } c \not\equiv m (\mod 2), \\ 0 & \text{otherwise,} \end{cases},$$

so r^* will guarantee that $\kappa q+2r+r^*$ can satisfy any remainder in $\{0,\ldots,\kappa(\kappa-1)\}$. If κ is odd, then q and r can be found such that $\kappa q+2r$ can be to equal any amount of degrees remaining, and if κ is even, then the parities of m and c must differ for the remaining degrees to be odd. Note r^* will force q and r to satisfy the equation.

In Case 2, when $(\delta+1-\kappa+m+c)(\kappa-m-c) < \kappa(\Delta-\delta)$, there are not enough edges that can be added to H to satisfy degree Δ. After attaching $(\delta+1-\kappa)\kappa+(\delta+1-\kappa+m+c)(\kappa-m-c)$ edges from L and M,

$$\kappa(\kappa-1) < \kappa\Delta-(\delta+1-\kappa)\kappa-(\delta+1-\kappa+m+c)(\kappa-m-c).$$

In this case, we attach L to H in the same way, and fix our cut vertices as a complete graph. To give each vertex in H degree Δ, $|[H,L]|=\kappa(\Delta-\delta)$ since for any $s_k\in H$, $\rho(s_k)=\kappa-1+\delta+1-\kappa=\delta$. Since $(\kappa-m-c)(\delta+1-\kappa+m+c) < \kappa(\Delta-\delta)$, there are sufficient edges so that each vertex in L has degree δ. Hence, we will prove that we can satisfy the maximum degree in H by showing that by attaching $\kappa(\Delta-\delta)$ edges evenly over M to H, for any $v_j\in M$, we have $\rho(v_j)\le\Delta$.

Proof. The maximum number of edges we can connect from H to L to ensure that $\rho(v_j)\le\Delta$ is $(\delta+1-\kappa+m+c)(\Delta+\kappa-\delta-1-m-c)$. The set of connecting edges of size $|[H,L]|=\kappa(\Delta-\delta)$, must be less than this. To show that

$$(\delta+1-\kappa+m+c)(\Delta+\kappa-\delta-1-m-c)\ge\kappa(\Delta-\delta),$$

we use the fact that $0<m+c\le\kappa-1$. It follows that

$$(\delta+1-\kappa+m+c)(\Delta-\delta)\ge\kappa(\Delta-\delta),$$

$$\delta+1-\kappa+m+c\ge\kappa,$$

$$\delta+1-\kappa\ge\kappa,$$

$$\delta+1-2\kappa\ge 0,$$

$$2(\delta+1-\kappa)\ge\delta+1.$$

This must follow since by Theorem 3.4,

$$2(\delta+1-\kappa)>\Delta\ge\delta+1.$$

□

Thus each cut vertex in H will be given degree Δ, and for each $v_j\in M$, $\delta\le\rho(v_j)\le\Delta$.

Example: Realizing $(3,9,9,12)$ Minimally

Given the parameters $(3,9,9,12)$, we can see that with $\Delta = 12$,

$$2(\delta+1-\kappa) = 14 > \Delta \geq 10 = \frac{\delta}{\kappa} + 1 + \delta - \kappa.$$

By Theorem 7.3, the minimum size is $n = 2\delta + 2 - \kappa + m$ where m is defined by

$$m = \left\lceil \frac{\delta + 1 + \sqrt{(\delta+1)^2 - 4\kappa(\Delta + \kappa - \delta - 1)}}{2} \right\rceil - \delta + \kappa - 1,$$

which gives us $m = 2$ and our minimum $n = 19$. To determine q and r we can solve the equation

$$\kappa\Delta - (\delta + 1 - \kappa)\kappa - (\delta + 1 - \kappa + m + c)(\kappa - m - c) = \kappa q + 2r + r^*$$

and find that $q = 2$ and $r = 0$. Therefore, we realize $(3,9,9,9)$ minimally as

$$K_7 \equiv H_{3,2} \equiv K_9,$$

which is the graph given in Figure 2.

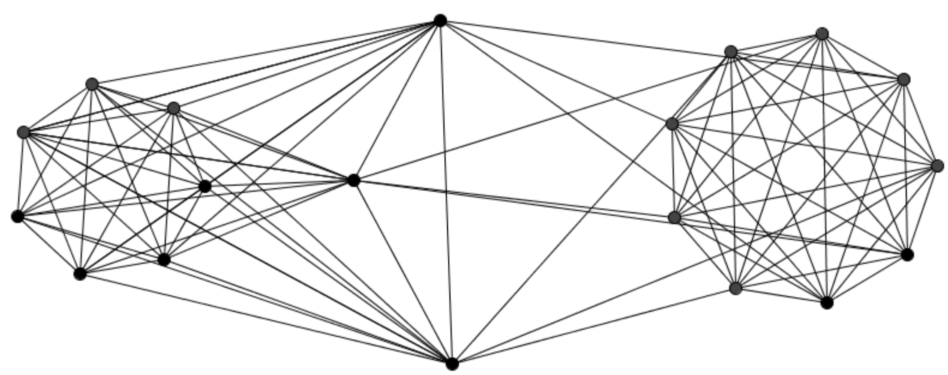

Figure 2

8 $\Delta < \frac{\delta}{\kappa} + 1 + \delta - \kappa$: No realizations smaller than $n = 2\delta + 2$

For $\Delta < \frac{\delta}{\kappa} + 1 + \delta - \kappa$, we can only realize our graph for $n \geq 2\delta + 2$. To realize $(\kappa, \delta, \delta, \Delta)$ that satisfy this inequality we must return to Section 1. We will now show that this inequality provides us with no minimum realization.

Theorem 8.1. ***Given*** $(\kappa, \delta, \delta, \Delta)$ ***that satisfy*** $\Delta < \frac{\delta}{\kappa} + 1 + \delta - \kappa$***, the smallest realization is*** $n = 2\delta + 2$***.***

Proof. Assume that we are given $\Delta < \frac{\delta}{\kappa} + 1 + \delta - \kappa$. We will show that there is no realization with $n < 2\delta + 2$. To do this we will prove that for any $(\kappa, \delta, \delta, \Delta)$ that satisfy the stated inequality, it is impossible to realize the parameters for $n = 2\delta + 1$. If it were possible to realize $n = 2\delta + 1$, then we could realize our graph in the form:

$$K_{\delta+1-\kappa} \equiv H^r_{\kappa,q} \equiv K_\delta,$$

since the smallest realizations are of this form, by Theorem 7.2. To attach edges to our cut vertices, $H^r_{\kappa,q}$, we notice that if $q = r = 0$, then the maximum number of edges equally connected to the cut vertices, $\sum_{s\in S} \rho(s) \leq \kappa\Delta$. Using the inequality with our parameters we can rewrite this as $\sum_{s\in S} \rho(s) < \delta+\kappa+\kappa\delta-\kappa^2$. This means that the maximum number of the edges needed to connect both L and R must be less than $\delta + \kappa + \kappa\delta - \kappa^2$. Since our minimum degree of the graph is δ, each $u \in L$ needs κ edges since they have degree $\delta - \kappa$ and each $v \in M$ needs 1 edge since they each have degree $\delta - 1$. Therefore, the total number of edges to guarantee that for all $w \in (L \cup M)$, $\rho(w) = \delta$, there must be a total of $(\delta + 1 - \kappa)\kappa + \delta$ edges. This simplifies to $\kappa\delta + \kappa - \kappa^2 + \delta$ which is a contradiction since $\sum_{s\in H} \rho(s) < \delta + \kappa + \kappa\delta - \kappa^2$. This means that to connect L and M to H to guarantee that all vertices have degree δ, there must be $s \in H$ where $\rho(s) > \Delta$. Therefore, we cannot realize a graph of size $n < 2\delta + 2$ given parameters that satisfy $\Delta < \frac{\delta}{\kappa} + 1 + \delta - \kappa$. $\square$

9 Conclusion

In conclusion, for $\kappa > 1$ and $\Delta + \kappa < 2\delta$, we can realize $(\kappa, \delta, \delta, \Delta)$ for any given parameters excluding $(2, \delta, \delta, \delta)$ where δ is odd. For any parameters we can find the minimum realization to be $n = 2\delta + 2 - \kappa + m$, for some $m \in \{0, \ldots, \kappa\}$, and given such m we can realize any $n \geq 2\delta + 2 - \kappa + m$. This is the fourth and final thesis that concludes Professor Wayne M. Dymacek's research project *Realizability of n-Vertex Graphs with Prescribed Vertex Connectivity, Edge Connectivity, Minimum Degree, and Maximum Degree.* With the completion of this project, working through hundreds of cases, Professor Dymacek's students have successfully completed an exhaustive system to determine the realizability of any given parameters and produce these simple and undirected graphs for any possible order that is desired. The complete project can be attributed to the hard work by the following graduates and faculty of Washington and Lee University:

- Dr. Wayne M. Dymacek, Professor of Mathematics
- Louis Joseph Steiner, Class of 2008
- Alyssa P. Hardnett, Class of 2014
- Candace Bethea, Class of 2015

With this complete system of algorithms, our only future work is looking for possible connections between distinct cases to simplify the extensive nature of this project.

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