Mathematical Ontology and Epistemology:
An Analysis of Quine's and Maddy's Unique Arguments

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Contents

I. Introduction

This thesis is concerned with the existence and knowledge of mathematical objects within a naturalistic philosophy. I will begin with a brief explanation of what constitutes a mathematical object and will then define and briefly analyze the two most extreme arguments for their existence, pure Platonism and pure concretism, using Plato's and Mill's views, respectively. I will then explain the difficulties for any argument of this type that arises with Benacerraf's dilemma: that any argument regarding mathematical ontology and epistemology must provide a plausible account for both mathematical truth and knowledge in a way that does not differ from our accounts for truth and knowledge of anything else.

I will then consider the Quine-Putnam Indispensability Argument, which provides arguably the best known answer to Benacerraf, beginning first with an explanation of Quine's empirical holism and resulting scientific naturalism, which is key to understanding his overall argument. This will follow with an outline of fundamental objections by Penelope Maddy that stem from differences between her and Quine's naturalism.

I will then review areas of general concern for naturalistic philosophy and how both Maddy and Quine overcome them. I will follow with an attempt to individually respond to Maddy's objections of Quine's Indispensability Argument using the foundations of his empiricism and naturalism, arguing that Maddy's counter-examples are times when either scientists were wrong or they demonstrated a need to further constrict what constitutes our best scientific theories. I shall also discuss how the differences in mathematical and scientific goals should not concern our philosophical inquiries.

Given that we have good reason to have ontological commitment to sets, I will review Maddy's set theoretic realism and her modifications of Benacerraf's definition of knowledge, and

I will highlight areas of concern and inconsistencies in her argument. Lastly, I will discuss any final consequences of these arguments, as well as any open questions.

II. What are Mathematical Objects?

There is a debate within the philosophy of mathematics on the quality of mathematical objects and whether they exist abstractly, outside the physical realm, or concretely, subject to empirical analysis. Geometric objects, such as circles, triangles, spheres, and polyhedral, as well as algebraic objects, such as groups and rings, are all candidates for qualification; but, the most theoretically contemplated "object" by non-mathematicians is generally the natural numbers. However, according to basic number theory,[1] the natural numbers are defined as a scaled collection of sets such that a succeeding number is the set containing the previous number,[2] with "0" defined as the empty set.[3] This is one of the basic results of Zermelo-Fraenkel set theory (ZF), which along with the axiom of choice[4] (C, together denoted ZFC), is considered the founding theory of our current application of set theory in mathematics. This definitional reduction of a proposed mathematical entity to set theory is not unique to the natural numbers (discussed later in detail). Consequently, as sets seem to play a pivotal role in mathematics, for the purposes of this paper we will be examining the existence of basic sets as our primary candidate for an existing mathematical object.

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[1] References to number theory in this thesis should be understood with the earlier philosophical definitions of the subject, thus, are more relatable to arithmetic than the mathematical definition.

[2] The number 0 is the empty set: $0 = \varnothing$. The number 1 is the set containing 0: $1 = \{0\} = \{\varnothing\}$. The number 2 is the set containing the set 1 and 0: $2 = \{0,1\} = \{\varnothing,\{\varnothing\}\}$. The number 3 is the set containing sets 2, 1, and 0: $3 = \{0,1,2\} = \{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}$, etc. Thus, numbers can be thought of a special set that is a specific collection of sets within sets.

[3] Gregory, "Set Theory [Forthcoming]."

[4] The Axiom of Choice states (informally) that given a collection of non-empty sets, one can create a set containing a specific element from each.

III. The Extreme Views: Pure Platonism and Pure Concretism

The explanation of how we come to know mathematical objects, such as sets, and the realm in which they exist can generally be placed between two extreme arguments: pure Platonism and pure concretism.

A pure Platonistic view holds that mathematical objects exist perfectly among Plato's abstract forms, which only philosophers, and in this case mathematical philosophers, can come to understand through intensive study. A form is an abstract and perfect model of an object, existing outside the spatiotemporal realm, which defines all the object's properties from which concrete universals can be defined.

According to Aristotle in Book XIII and XIV of his *Metaphysics*, Plato maintained that there are two types of numbers, idealized and mathematical.[5] Idealized numbers are our imperfect concepts of numbers and exist concretely, while mathematical numbers are the perfect, abstract form – the actual mathematical object to which we refer when discussing numbers. I will not entertain a complete explanation of Plato's argument from *The Republic*, but these forms can never be fully understood empirically, as all empirical data is derived from idealized objects, which are imperfect representations of the forms themselves. Rather, the pure Platonist adopts a form of rationalism, relying on philosophy and a priori conclusions to understand these mathematical abstract objects. To modernize Plato's views, I claim that given basic number theory, Plato's forms of numbers would present themselves as forms of special sets. While Plato claimed that observable, idealized objects were governed by arithmetic, and mathematical objects, by geometry, some later philosophers disagree on this classification due to an advanced interpretation of set theory.

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[5] Wedberg, *Plato's Philosophy of Mathematics*, 9–20.

However, I still classify them as pure Platonists as long as they classify mathematical objects as abstract and base their knowledge of a mathematical statement on the a priori facts of its structure.

Unlike a Platonist, a pure concretist would deny any reference to a priori mathematical knowledge, and rather would claim that we understand mathematics through empirical inquiry. Thus, our knowledge of the truth value of a mathematical statement is based solely on our observations of previous mathematical statements and values, implying that a statement such as $2 + 1 = 3$ is known to be true only because we have always observed it to be so in the physical realm. One of the greatest proponent of mathematical concretism is John Stuart Mill. Mill asserts that numbers are based on definitions, which are based on an observed, fixed fact.[6] Thus, calculations do not follow from the definition themselves but from an observed matter of fact. Mill defines numbers as a collection of objects (a set, though he doesn't use this terminology) which appear to be a collection of individual objects, yet cannot be split up into smaller parts.[7] Furthermore, he claims these sets do in fact exist concretely as physical collections of objects rather than abstract forms.

According to Mill, we may make an inductive law that allows us to apply our observations of mathematical calculations for small, observable numbers to larger, less observable calculations (such as $1,000,000 = 999,999 + 1$). However, an absolutely pure concretist would argue that such calculations must be observed directly to allow us to determine if it is in fact true.

Both these views present initial strengths and weaknesses, mainly due to epistemological concerns. Pure Platonism presents the abstract view of mathematics which most mathematicians seem to employ (mathematical objects exist, but not concretely); however, the prescribed

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[6] Mill, *System of Logic*, 292–300.
[7] An interesting result is that under this view, 0 is only a constructed symbol that has no meaning and acts as only a symbol of a physical, observable process which relates two sets, not unlike addition and other functions.

connection to its mathematical truths (abstract forms) through philosophical inquiry seems dubious and is unlike the proofs derived by mathematicians as mathematical justification.

Furthermore, throughout the history of mathematics, there have been several direct and fundamental axioms that are clearly based on empirical evidence. Hilary Putnan details several of these occurrences, one example being Descartes' original justification for one-to-one correspondence from the points of a straight line to the real numbers. His proof was founded in geometric observations rather than a purely abstract construction of a mapping from the reals to the rationals.[8] Though number and set theory would eventually provide the definition of a number that allowed for this abstract construction, the theorem was already accepted as an extremely applicable mathematical truth that was used to explain numerous physical and mathematical realities prior to abstract explanation. Several similar examples, such as Zermelo's Axiom of Choice, can also be found as well.

Alternatively, pure concretism lacks the dubious connection with its mathematical truths (concrete objects); however, its naïve empirical foundations seem quite contrary to mathematical practice. Many mathematical statements are made through inductive arguments; however, pure concretism denies such an argument, as actual empirical evidence is needed for every conclusion. Furthermore, as with any empirical conclusion, even if we make conclusions based only on previously observed evidence, such conclusions can never be absolutely known as True, as we can never have complete knowledge of the entire spatiotemporal universe to rule out all other possibilities and competitors we may have yet to observe. This seems contrary to our general view of mathematics as one of the most certain forms of knowledge, so most mathematicians would generally regard the concretist epistemology with extreme distaste.

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[8] Putnam, "What Is Mathematical Truth?"

IV. Benacerraf's Dilemma

In a more general approach, Baruj Benacerraf denounces both pure Platonism and mathematical formalism[9] (the claim that logical deduction through proofs provide sufficient connection to truth) by exposing their epistemological weaknesses and outlining the qualities of acceptable philosophical theories of mathematics. He begins with outlining the two conditions any theory must satisfy: First, it should not deviate dramatically from our standard theory of truth/semantics. This condition is required because Benacerraf does not want to grant a special justification method for mathematics; we want our account of mathematical truth to be justified the same way we account for the truth of any other thing. Second, supposing that there exist some mathematical truths and we have knowledge of some of them, any theory proposed must also provide a plausible account of this mathematical knowledge.[10]

By "an account of mathematical truth," I mean provide clearly what constitutes our mathematics and how we render our mathematical statements true or false (i.e. to what are we really referring when we use sets or numbers in mathematics). By "an account of mathematical knowledge," I mean provide a plausible account of our knowledge of mathematical claims, which to Benacerraf, is one which is causally based.

Platonism provides a great account of mathematical truth for it argues that our true mathematical statements are true by nature of the existence of their corresponding mathematical objects and the abstract relations which exist between them. Furthermore, this account is clearly made within our general semantics as we refer to the existence of either concrete examples or

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[9] I reference formalism to highlight the issues of claiming our mathematical proofs are sufficient to provide an overall ontological/epistemological theory, which may seem a natural argument to practicing mathematicians.

[10] Benacerraf, "Mathematical Truth."

abstract definitions when determining the truth value of any non-mathematic statement. Thus, Platonism only requires a good account for mathematical knowledge to pass Benacerraf's conditions; however, it fails this final condition. As Platonism holds that mathematical objects exist abstractly, we can never have causal interaction with these objects; thus, it can not provide an accurate account of knowledge.

Formalism, on the other hand, has a great account of mathematical knowledge for it argues that such knowledge is obtained by providing a formal proof of any mathematical statement. As any mathematical proof can very clearly be linked back to underlying founding axioms of mathematics, of which we have fundamental knowledge, and as such proof is tested and verified universally by other mathematicians, we certainly have strong knowledge of these mathematical statements. Formalism, however, fails to have an acceptable account of truth, as it claims that if we can provide a proof of a mathematical statement, we have also shown it to be true. First of all, this account fails to conform to our universal theory of truth (i.e. ordinary semantics), as we don't rely solely on formal proofs to account for our non-mathematical truths. Secondly, it is clearly wrong due to the consequences of Gödel's Incompleteness Theorem, which states that no finite set of underlying axioms of number theory can ever be both consistent and complete. Thus, our underlying axioms of mathematics in which our formal proofs take root are either not complete or are contradicting, which clearly implies they cannot accurately account for truth.

Benacerraf's dismissal of both pure Platonism and formalism, two very popular arguments, highlights the difficult challenge in providing both an account of mathematical knowledge and account of truth in a way that is consistent with our ontological and epistemological justification for anything else. We shall therefore consider Quine's theory, which claims to succeed in doing so.

V. The Cornerstone of Quine's Theory: Empirical Holism

Quine's proposed theory (the Indispensability Argument) is an indirect application of his theory of empirical holism; thus, to appreciate his final argument, one must first understand his justification for the necessity for such a theory. This can be found in his dismissal of reductionism and of the relevance of the analytic-synthetic distinction.

Analytic statements are those which ground their truth in the statement itself (ie. they are allegedly true by virtue of meaning). The classic example – "All bachelors are unmarried" – is analytic because the definition of a bachelor is "an unmarried man;" therefore, the statement is true as the meaning of bachelor inherently encompasses a state of unmarriedness. If one were to instead say "All bachelors are under 130 years old," this statement would be synthetic because even though it is true, the quality of being under 130 is not inherent in the definition of a bachelor.

Rudolf Carnap, against who's views Quine is responding, argued that any language (ordinary or hypothetical) can be constructed such that what is analytic in one is synthetic in another.[11] Essentially, if one were to take the intersection of analytic statements in all possible languages, there is no statement that would be returned as analytic in all of them. Thus, to answer metaphysical questions, Carnap argued one must first determine the framework of the language in which we address it. By examining our language and the axioms on which the framework is founded, we can determine if it is a question worthy of empirical inquiry through science, or if it is one which rather addresses the linguistic framework of the language employed. In the case of the latter, the question is just a pragmatic one concerning the choice of framework; however, if the former, these inquiries can be answered through verifiable scientific theories which employ such framework.

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[11] Carnap and Hart, "Empiricism, Semantics, and Ontology."

Carnap's argument is based on the premise that there is a clear distinction between analytic and synthetic statements and that such a distinction is available in all useful languages. In "Two Dogmas of Empiricism," Quine rejects the power of this distinction by connecting it with and rejecting reductionism, the argument that a proposition's meaning is the basis on which it is verified. Quine links these pillars of classical empiricism by showing that those statements that are analytic have no interesting meaning, for they are self-verifiable, while those that are synthetic do possess such meaning. Quine's main concern arises with how the self-verifiable, analytic "meaning" is transferred between seemingly identical statements that differ only by synonyms, such as "No unmarried man is married" and "No bachelor is married." The transfer of this analyticity is also a question for mathematics, for how do we know that two seemingly identical mathematical statements hold the same meaning.[12]

One possibility proposed is that synonyms can be explained through definitions; however, definitions presuppose synonymy and do not explain the link between their meanings; thus such an answer is insufficient. Additionally, the possibility of such words being interchangeable *salva veritate* (with unharmed truth) is also dismissed, for if synonyms are interchangeable, then the language must be free of homonyms. Otherwise, words such as "bachelor" could refer to something else, such as a bachelor of arts, and therefore the interchangeability is true only under necessary conditions. As the only "necessary" statements are ones which are analytic (ie. *salva analyticitate*), this argument is circular and thus unacceptable.

The two remaining possible explanations, references to state descriptions and artificial languages, are both dismissed through reference to Quine's arguments in his paper "Truth by Convention." Carnap's theory of state descriptions is originally outlined in his argument for modal

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[12] For example, do these two statements truly mean the same: "Pi is an irrational" and "The circumference of a circle divided by twice its radius is an irrational."

logic and essentially argues that if we take two analytic sentences and break them down into atomic, logical statements, and if all these atomic statements have the same truth value, then we have shown synonymy.[13] Carnap also suggests a more general argument: we can construct any artificial language, not just one based in logic (such as mathematics), that is acceptable as long as the axioms on which the linguistic framework is developed provide for a clear definition of synonymy.

Both these arguments rest in the premise that we can break down our (mathematical) language to a finite set of analytic axioms (like the formalist), whether it be logic or a set of artificial reference statements. Such analytic axioms in an artificial language would be true by its framework, and as there exists a systematic, finite list of axioms on which logic is founded, it also appears that logic is in fact analytic (ie. true by convention). However, though it may take some work, it is also possible to explain our scientific truths through a similar axiomization which is also finite; thus, to be consistent, we must also conclude in their analyticity. This concern, however, is countered by Carnap who attempts to claim that we distinguish between analytic and non-analytic scientifically verifiable statements by determining which statements are fundamental to the entire paradigm and which have the potential for future change. Those deemed fundamental are known to be analytic, while those that are not, non-analytic.

Quine argues, however, that such methodology in regard to scientific classification is rather only a convenience spanned from a natural human desire to make as few changes as possible to already accepted theories when new information is discovered. Thus, there is no underlying substance to Carnap's categorization. This is especially true for mathematical and logical truths, which are really only deemed analytic because they are so integrated in the rest of our scientific inquiries: changing them is possible, but to do so may alter our entire scientific paradigm,

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[13] Carnap, *Meaning and Necessity - A Study in Semantics and Modal Logic*.

something we naturally desire to avoid. Thus, we cannot explain synonymy through state descriptions nor through reference to artificial languages. As a result, Quine concludes that one cannot interestingly distinguish between analytic and synthetic statements made through any language (as ours is based in logic and any artificial language is also dismissible). The only distinction we can make is made through a reference to the linguistic framework, which is an uninteresting, pragmatic question.[14] Furthermore, the reductionist argument is also irrelevant through its intertwined relationship with the dismissed analytic-synthetic distinction.

As there is no interesting answer to be received from asking why individual statements of a language are analytic versus synthetic, Quine argues that we must therefore base our verification of proposed statements (such as those derived from empirical, scientific hypotheses) by considering only the theory as a whole, rather than analyzing the truth values of its individual parts. If in our empirical testing we find that our proposed theory fails, then we know that at least one of the initial assumptions (axioms) must be false, but we can never know with certainty which one. If, rather, we find through our repeated testing that the theory continues to hold, then we can continue to assume its verification. We can never know with certainty that the theory is True; though, we will continue to accept it in the interim, altering less fundamental theories until a new paradigm must be constructed based on new empirical evidence.

This assertion of a scope of knowledge limited to empirical inquiry is what leads Quine to dismiss any metaphysical or a priori inquiries or conclusions and recognize that our knowledge can only be made from inquiries into the natural world. So, how do we make such inquiries in a way that is acceptable to Quine's empirical philosophy? Quine's answer is science. The scientific method and the contemporary, empirical philosophy of science, argued by the falsificationism of

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[14] Thus any formalist argument to mathematical knowledge or truth is dismissible.

Karl Popper, never grants "Truth."[15] It recognizes the limitations of empirical research yet also the advantages of continuous verification of applied theories. Furthermore, our empirical results of science are the consequences of theories applied as a whole, rather than by an individual application of theoretic axioms, and thus our theories, as a whole, remain applicable until a new theory improves upon it. As a result, scientific inquiry can support Quine's empirical, holistic philosophy, and its results are thus admissible as an explanation of our philosophical questions. This relationship is what Quine refers to as "the continuity of philosophy and science" and is what forms his unique scientific naturalism.[16]

VI. The Quine-Putnam Indispensability Argument

Our question now is how does this relationship affect our philosophical ontological commitments? In his paper "On What There Is," Quine argues that the language we employ (mathematical, scientific, semantic, hypothetical, etc.) is what determines the ontological commitments we must make. Thus, if such a language allows reference to any entity, we must grant ontological commitment to that entity. Note, however, that while we may have commitment to the existence of such entities, this does not mean that this ontology automatically represents the truth. For example, some hypothetical language may allow references to mythical creatures or to "the greatest prime number,"[17] but obliviously we don't want to be committed to these sorts of entities. Rather, we must only be concerned with the ontological commitments given by the language which best represents our understanding of the world. As we have determined that science represents this

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[15] Popper, "Science as Falsification."

[16] Quine, "Naturalism: Or, Living Within One's Means," 256–7.

[17] Euclid proved that there are infinitely many primes in 300 B.C.E. See Euclid and Heath, *The Thirteen Books of Euclid's Elements*.

best language for empirical and philosophical inquiry, we are therefore committed to all and only the entities to which we find ourselves referring through science.

So to what entities are we committed through science? Well science is a very dynamic field with new theories, hypothesis, and amended conclusions proposed and tested daily by scientists around the world. Obviously, we shouldn't be immediately committed to entities of any scientific theory, as some theories, while scientific, may not be uniform in scientific methodology; rather, we must be only concerned with determining our commitment to the entities which are referenced through our best scientific theories. Quine never directly defines what he means by best; however, in defending the strength of molecular theory in his article "Posits and Reality," he gives us a good outline of the types of qualities for which he is looking for:

> One is **simplicity**: empirical laws concerning seemingly dissimilar phenomena are integrated into a compact and unitary theory. Another is **familiarity of principle**: the already familiar laws of motion are made to serve where independent laws would otherwise have been needed. A third is **scope**: the resulting unitary theory implies a wider array of testable consequences than any likely accumulation of separate laws would have implied. A fourth is **fecundity**: successful further extensions of theory are expedited. The fifth goes without saying: such testable consequences of the theory as **have been tested** have turned out well, aside from such sparse exceptions as may in good conscience be chalked up to unexplained interferences.[18]

Given all of our best theories, there are always integrated theoretic parts which appear in all of them. An un-controversial example of such a recurring "part" would be "we can make references to physical objects which we collectively sense and perceive." Because no best scientific theory is without these recurring "parts" we can call them "indispensable" to science. Thus we have ontological commitment to physical objects, and we can be philosophically satisfied with this commitment because it is indispensable to our best science as a whole.

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[18] Quine, "Posits and Reality," 234. These were the qualifications that permitted the acceptance of more observable theories, but Quine does not differentiate between levels of observation in his qualifications of best theories.

So what else is indispensable to our best science? Well clearly mathematics is a very integrated part of any scientific theory: physics is essentially applied mathematics and we use physics to unify and justify conclusions from all other scientific disciplines. It may not seem apparent at first that mathematics plays a role in theories such as the geological conclusion that "Hypabyssal igneous rocks are formed due to cooling and solidification of magma just below the surface,"[19] but when we assess the empirical justification for such theories, we always find either inter-disciplinary references (such as chemical formulas and resulting physical interactions) or just straight mathematically-rich comparisons of observations (such as comparisons of percentages, ordering, volume, size, etc.) that highlight the integration of mathematics. No scientific theory can escape this tight integration of mathematics in science; thus, we can conclude that mathematics is an indispensable part of our overall scientific theory. Therefore, we have ontological commitment to any and all of the mathematical entities which are referenced by science. This argument is what is known as the Quine-Putnam Indispensability Argument, which is here formalized for clarity:

1. We ought to have ontological commitment to all and only the entities

    that are indispensable to our best scientific theories.

2. Mathematical entities are inherent to mathematics which is indispensable our best

    scientific theories

➤ We ought to have ontological commitment to mathematical entities.

VII. Referencing Mathematical Objects

Now we must ask to what mathematical objects do scientific theories refer? As discussed in Section I, it seems that there are many candidates for such objects, but it seems that sets are the

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[19] Best, *Igneous and Metamorphic Petrology*, 1.

most basic and well accepted. Additionally, it seems that set theory is itself an indispensable part of mathematics as from sets we can understand numbers as well as relationships between any mathematical symbol (numbers, variables, etc.). [20] In *The Foundations of Mathematics*, Kenneth Kunen discusses the "all-importance" of set-theory, even arguing that "all abstract mathematical concepts are set-theoretic…[and] all concrete mathematical objects are specific sets." (I don't necessarily agree, however.) Furthermore, when other mathematical disciplines encounter open questions, the general tendency is to refer to set theory as the underlying mathematical framework.

Regardless of the indispensability of sets within mathematics itself or whether the only mathematical objects are sets, it is clear that theories of mathematics and science regularly make reference to them. This reference is both abstract and concrete: "the set of the empty set" is an obvious abstract reference which is used frequently in the construction of numbers (see Section I), while "the set of four books" is a more concrete reference.[21]

As mathematics is indispensable to science, it would follow from Quine that sets do in fact exist, including those which are abstract. (Note that I am not arguing sets are the only mathematical object to which we have ontological commitment.) At face value, this seems a peculiar conclusion from Quine, who is generally hesitant to admit to the existence of abstract objects; yet, it is the ultimate conclusion he must make in order to fit his arguments for empirical holism and his scientific naturalism.

Returning to Benacerraf, it seems that Quine has successfully overcome his dilemma. He provides a good account for mathematical truth: like Platonism, our mathematical statements are true by the nature of the existence of their corresponding mathematical objects and the abstract

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[20] Penelope Maddy has an entire chapter labeled "Set Theory as a Foundation [of Mathematics]" in her 1997 book *Naturalism in Mathematics*.

[21] At least it's elements are concrete; whether the set itself is concrete is up for debate (Sections XII-XIII).

relations which exist between them. This is similar to how we account for the truth of anything else – through existing concrete objects or abstract definitions of which we gain knowledge through science. He also provides a good account of knowledge: we have knowledge of mathematical statements just like we have knowledge of anything else: they are an integrated part of all our scientific theories, which provide the best account of knowledge of the world. As a result, Quine's argument is still referenced as the strongest for the existence of mathematical objects.

Penelope Maddy offers the strongest rebuttal to Quine's Indispensability argument. Her arguments are based on the observation that scientists and mathematicians do not in practice employ Quine's holism, nor do they respect the ontology to which they commit us as they use any mathematics necessary to achieve approximations or their desired results. Furthermore, Maddy argues that Quine's naturalism does not respect the methodology of mathematics, as it suggests that any non-empirically justifiable mathematics is just recreational. We shall explore these objections in detail and discuss the implications of Maddy's proposed mathematical naturalism, which arguably is more faithful to the methodology of practicing mathematicians. Note that in this thesis the methodology or practice of mathematicians or scientists is a reference to the actual methods employed by the practitioners, whereas the ideals of science or mathematics, or references to overall science or mathematics itself, are references to the idealized and generalized methods that are derived directly from the overall goals of the subjects. This is an acute distinction, but we could also explain it as approaching the disciplines from a bottom-up versus top-down perspective, respectively. An example of the latter would be "science requires continuous verification of theories," while the former refers to how that verification is actually executed by scientists.

VIII(a). Maddy's Objection: Practicing Scientists Do Not Employ Quine's Holism

Recall that Quine's holism leads him to conclude that we are ontologically committed to any and all entities that are indispensable to our best scientific theories. Maddy's first objection points out that in some cases, scientists do not in fact believe in the existence of the entities that play a critical role in our science.[22] Maddy references the discovery of the atom at the turn of the twentieth century. After years of empirical tests and theories from scientists such as Dalton, Gay-Lussac, and Dumas, when Cannizzaro presented his culminating and verifiable theory that distinguished between the atom and the molecule, it took scientists almost half a decade to finally agree on the existence of atoms. Even though Cannizzaro's research followed all of Quine's arguments for acceptable molecular theory (simplicity, principle familiarity, scope, possibility for future growth, and continued verification of the theory), many scientists met it with skepticism and rather claimed that until the theory passed a further, direct test, the atom should only be considered a "useful fiction." Soon after Cannizzaro's research, scientists agreed in the 1860's that the atom, regardless of its disputed existence, was to be "the fundamental unit of chemistry." Regardless of the apparent indispensability of atomic theory, the ontological debate continued until the early twentieth century, when the general consensus of scientists eventually agreed on the atom's existence. Clearly, Maddy argues, this raises the concern that while Quine grants the existence of all indispensable entities of our best theories due to empirical holism, practicing scientists do not necessarily provide such a commitment.

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[22] Maddy, *Naturalism in Mathematics*, 135–43.

VIII(b). Maddy's Objection: Scientists Disregard Mathematical Ontology

Now that we have reason to believe that not all entities of our best theories deserve ontological commitment, we must explore whether our mathematical entities are deserving. However, it seems that many of the mathematical applications in our science are based on idealized conditions which scientists agree do not actually occur. Examples of this can be seen in equations such as those that calculate a lack of sound waves traveling through a vacuum, while no true vacuum exists; or in the calculation of fluid dynamics based on "ideal" fluids; or the calculation of short distances on the Earth's surface using lines rather than arcs (as the Earth is curved). Scientists all use these equations to get a very accurate estimation of actual events; however, these all remain only estimations, thus they can not force us to have ontological commitments as scientists agree that they are not completely true.[23]

While Quine acknowledges that there certainly exists a wide range of idealizations in science, he argues that the behavior of actual situations as the conditions approach idealization is what is shown to be true, not the idealization itself.[24] In other words, speeds of objects calculated using a frictionless plane (an idealization) is based on the behavior of the speeds of actual objects as actual planes have less and less friction. Thus, idealizations are based on empirically verifiable results and thus carry ontological commitment.

Maddy dismisses this claim by pointing out that there exist several idealizations in science which cannot be based on the convergence of actual conditions to ideal. One example is ideal fluids, for as fluids approach idealized conditions, their molecules loose their liquid properties and assume the form of a solid.

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[23] Ibid., 143–52.
[24] Quine, *Word and Object*, 250.

One may dismiss these initial idealized concepts and claims, stating that surely our empirical observations of geometric objects (and the ratios/numbers that their properties yield), as well as the verifiable theories based on them, give us reason to have ontological commitment to such mathematical objects, for physicists use circles, squares, and polyhedral all the time! Note, that for these geometric objects to occur in the natural world, space-time must be continuous. It seems, however, that this is not the case,[25] but at the very least is still an open question.[26] As no current verifiable theory grants a perfect concrete shape from which we can obtain empirical evidence (as no line is truly straight, nor circle non-elliptical, etc.), our mathematical formulas, which are based on such perfection, are rather only further idealizations which do not warrant ontological commitment.

Furthermore, like geometric objects, it seems that scientists frequently utilize mathematics in science without regard to the ontology in which they ultimately commit themselves. Another good example of this trend is the practice of physicists actively using the real numbers throughout their theories; however, if space-time is in fact discrete, as many scientists believe, then it is impossible for non-whole fractions or irrationals to exist concretely.

Maddy concludes that it is the practice of scientists to either consciously use mathematics in ideal theories or utilize it without regard to the resulting ontological commitments. Thus, it seems that most, if not all, mathematical entities utilized by science do not warrant ontological commitment.

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[25] Even Einstein was unable to create a theory founded in the continuity of the universe. If a future theory succeeds, then this point may be subject to dismissal.

[26] Forrest, "Is Space-Time Discrete or Continuous?"

VIII(c). Maddy Obj.: Quine's Naturalism Does Not Respect Mathematical Methods

Maddy's final objection is concerned with the ontology and methodology of high-level mathematics.[27] First, it seems that Quine's naturalism limits mathematical ontology to its application in empirical science, essentially throwing out all of continuous mathematics. Maddy claims this philosophy is far too expensive and is quickly dismissed by the mathematical practitioner who cares not for empirical justification. Rather, mathematicians rely solely on the proper mathematical foundations on which it was built. Specifically, within mathematics, there exists many open questions in set theory which have been proven to be unable to be answered using current ZFC set theory; they can either be added onto ZFC as an additional axiom or dismissed. An example of such an additional axiom is the continuum hypothesis (CH), which states that there does not exist an infinite set of real numbers such that there does not exist an injective correspondence between it and the reals, or between it and the natural numbers.[28] To evaluate CH, Maddy evaluates an even bolder axiom, the axiom of constructability (i.e. $V = L$), which implies CH but dismisses ZF. $V = L$ implies that every set in the mathematical universe (V) is constructible (L), which means it can be wrote in terms of a simpler set. Essentially, it requires introspection into a mathematical universe very different than than our own.

In addressing this question, mathematicians are not restrained by the limitations of physical science, and when they attempt to address the question, they do not base their methodology in science, but rather on proper mathematics.[29] In *Naturalism in Mathematics*, Maddy drives home this point by presenting a very complex proof against $V = L$ solely by adhering to the actual practices of mathematicians – stating the proof in a natural language connected with informal logic and a

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[27] Maddy, *Naturalism in Mathematics*, 152–57.

[28] More simply, an infinite set with size bigger than the naturals but smaller than the reals.

[29] Proper mathematics is just the everyday formulas and general mathematics as views by a mathematician without regard to its scientific application.

strong understanding of current ZFC theory. While I do not posses the mathematical skill necessary to fully evaluate the proof presented, the proof, if sound, would demonstrate the ability of a mathematician to answer a purely mathematical question in a manner quite separate from anything even possibly related to science.

Similarly, it should be noted that all our mathematical principles utilized in our mathematical theories are never really tested empirically, as no non-mathematical language has been created to allow for its verification. Thus, mathematics fails to gain the verification we see applied to scientific theories in science. However, once again, this does not seem to concern the mathematician.

IX. Defining Maddy's Naturalism and Its Differences to Quine's

Before we discuss possible defenses of Quine's Indispensability, we should discuss the type of naturalism that is emerging from Maddy's argument and its philosophical consequences. The biggest difference between Maddy and Quine is the scope of their naturalism. Both concur that philosophy is continuous with science; however, the role of mathematics is incredibly different.

Indispensable Math in Best Science

Scientific Naturalism

Math
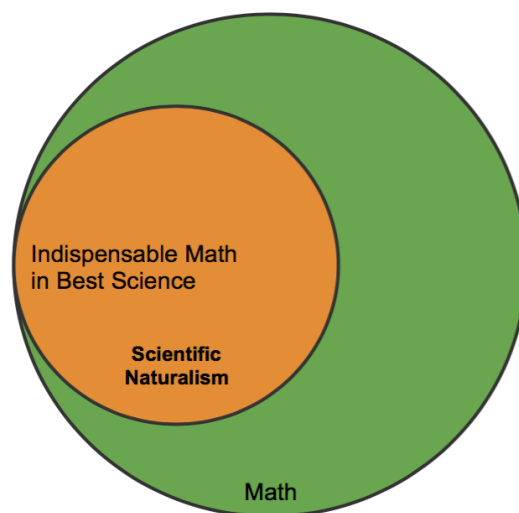
Quine's Naturalism

Quine's naturalism is purely scientific and is a top-down view based solely on the ideals of overall science, rather than the methodology. We know our world through science, and thus every entity on which our best scientific theories rely warrant ontological commitment. It just so happens that mathematics is an integrated part of our scientific theories, thus we can gain knowledge of the ontology of some mathematical entities. This theory rests on the premise of empirical holism: we must have ontological commitment to all of the entities of our best science, we can not pick and choose what we should believe – doing so would require some knowledge of which sub-theoretical statements are analytic vs synthetic, which Quine showed we cannot have. Consequently, Quine's scientific naturalism regarding "Indispensable Math in Best Science" is synonymous with ontological commitment to all entities within that category (the orange region). It is noteworthy that Quine leaves out many mathematical entities that are utilized in higher level set theory, as such theories are cases of "mathematical recreation" as they have no utilization in the empirical understanding of the world (the outer green region).[30]
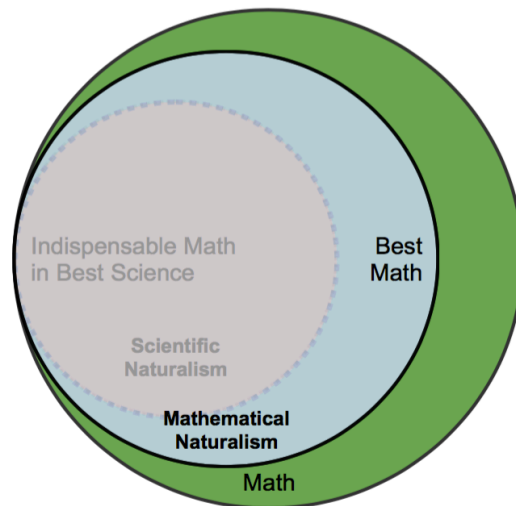
Maddy's Naturalism

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[30] Quine, *Word and Object*.

Maddy naturalism is much more complex as it is a bottom-up approach based in the methodology of science and math, rather than just the epistemological benefits of the ideals of overall science. It is important to note that Maddy does not disagree that mathematics is indispensable to our science, but as we've discussed, both have very different methodologies and goals. Recall that Quine argued that philosophy is continuous with science because the empirical practices of overall science are the best way to understand the world. Maddy agrees; however, because mathematics is an indispensable part of science but it has a different methodology, we must conclude that philosophy is continuous with science and mathematics. [31]

Maddy's overall naturalism is therefore split into two, scientific and mathematical. Scientific naturalism is confined to the evaluation of science by scientific methodology, while mathematical naturalism is confined to the evaluation of mathematics by mathematical methodology. An immediate consequence of this expansion is that we are now exposed to a wide variety of mathematical entities which are commonly referred to in our "best" mathematics (the blue overlay region – all mathematics without mathematical recreation), but not necessarily in empirical science. Note that many parts of our best mathematics are utilized by our science, but mathematics overall is not limited by it. Regardless, because both are continuous with philosophy, we should have reason to evaluate the ontological commitments for both categories. If we accepted empirical holism, we would be inclined to automatically commit to all entities of both naturalisms, but recall that Maddy dismisses this as unfaithful to methodology. Thus, scientists within their scientific naturalism have the freedom to make ontological commitments to whichever entities they feel are necessary for their best theories; they are not bound to every entity. Scientists, however, have no right to answer philosophical questions about mathematics in their own right, as such a

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[31] Quine says that we can answer all pragmatic and theoretical questions with science. Maddy says that we can answer some of these questions with science and some with mathematics.

question must be addressed within the broader mathematical naturalism by asking mathematicians which entities they feel require ontological commitment for their practices. Maddy is here proposing that the only ontological commitments we must make are to those entities which are deemed necessary by the practitioners of the original subject concerning that object; a stark difference from Quine, who grants it to all entities referenced by our best scientific language. Thus, Maddy would hold that Quine's scientific naturalism, when applied to mathematical philosophy, provides a severely limited, if not a pseudo-mathematical explanation, and cannot provide answers to our ontological inquiries.

So what entities are accepted by mathematicians in their methodology? Maddy has an unsatisfying answer: none.[32] Within mathematics it seems that mathematicians do not agree collectively on ontology, as some hold themselves as formalists, while others as logicists, institutionalists, predicativists, etc. but this does not affect their mathematical work in practice! All mathematicians build from fundamental axioms of their related mathematical field and agree that mathematical entities are indispensable to our best math, but there is not uniform ontological commitment. Note this is unlike scientists, who all agree in both the indispensability and existence of atoms in our best theories. This does not mean that mathematical entities do not exist, but rather that because we are restricted to mathematical methodology in our naturalism, we cannot make these commitments. It is important to note that Maddy is not really concerned with philosophical questions in mathematics; rather, she only wishes "to establish the legitimacy of the open questions of set theory," so this conclusion does not bother her.[33]

Returning to Benacerraf, mathematical naturalism seems to provide an acceptable answer to his dilemma. Recall that any theory proposed must provide both a uniform account of

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[32] Maddy, *Naturalism in Mathematics*, 201.
[33] Maddy, An Interview with Penelope Maddy.

mathematical truth and an account of mathematical knowledge, not just one of them. Mathematical naturalism is justified through our ordinary semantics/account of truth, as the mathematical language is one which is integrated into our philosophically "ordinary" scientific language and it too, must be considered ordinary. Specifically, it argues that our mathematical statements are true by the nature of the understood existence by mathematicians of their corresponding mathematical objects and the abstract relations which exist between them. Like the formalist, it accounts for mathematical knowledge by claiming that knowledge is obtained by providing a formal proof of any mathematical statement. It just so happens that there are no entities which have an "understood existence by mathematicians;" but this would not deter Benacerraf. It should be noted that Maddy's naturalism comes at a direct dismissal of Quine's account of mathematical knowledge. Quine claims we know mathematical statements like we know anything else – though our science. If Maddy's argument to the contrary is correct, it seems that Quine's theory is no longer acceptable to Benacerraf either.

X. Defending Naturalism

While there exist clear differences between Quine's and Maddy's naturalistic theories, we must first consider if their naturalistic approach is justified. There are several inherent issues that may arise in naturalistic theories, one of the greatest of which being the need for division between accepted best practices and meta-practices (e.g. science vs meta-science). A relevant question then to ask is how do we determine with either Maddy's or Quine's views what constitutes practicing mathematics or science, in contrast to meta-mathematics, meta-chemistry, meta-biology, etc. Ignoring Maddy's objections momentarily, Quine answers this concern by outlining what constitutes best scientific standards: simplicity, principle familiarity, scope, possibility for future growth, and continued verification of the theory. As a result, if a proposed scientific theory fails to

hold to one of these standards, then we can distinguish it as falling outside our best science. The non-best science would be scientific recreation. Likewise, the difference between science and meta-science is that the former is an attempt to explain reality with some respect for empirical methodology, while the latter lacks this respect or objective.

As Quine is focused solely on science, his distinction between our best mathematics and mathematical recreation mirrors the distinction between our best science and everything else. Thus, any further distinction between mathematical recreation and non-mathematics is not necessary for Quine's argument.

The distinctions necessary for Maddy are quite different. Maddy does not outline what constitutes our best scientific theories, though it seems that she does not disagree with Quine's, for she accepts them when she points to a "best" theory, molecular theory, in her argument against empirical holism. However, her use of Quine against himself may not serve as an endorsement of his definition. Regardless, the differentiation between best science and scientific recreation is not as concerning to Maddy, as she is solely interested in questions regarding the philosophy of mathematics, not of general science, which is more concerning to Quine. Her main concern is that by constricting ourselves to how science is practiced, we lose almost all of mathematics to mathematical recreation, as there is no scientific commitment to mathematical entities nor use in empirical science that is believed to represent reality.

Unlike Quine, Maddy argues that mathematical recreation is rather just the mathematics that is not our "best" mathematics. She seems to define our best mathematics (i.e. proper mathematics) by referencing the fundamental axioms of each mathematical discipline which are universally accepted by mathematicians. Any theory that can be proved from these axioms are considered best mathematics. Mathematical theories like those which assume ZFC +/- CH, which adds to ZFC set theory but cannot be proven or disproven within it, is classified as mathematical

recreation. Perhaps one day ZFC - CH will be considered a fundamental axiom, which is why mathematical recreation inquiring into the qualities of such a universe is still considered mathematics; it is just not the best, because it is not derived from the universally accepted mathematical axioms. Note that by Gödel's incompleteness, the finite set of accepted mathematical axioms must either be inconsistent or incomplete. This differentiation within mathematical methodology between best mathematics and mathematical recreation alleviates some of the worry that we are ignorantly stuck within a limited system, as we recognize through our pursuit of mathematical recreation that our best theories may be lacking. Pursuit of higher level mathematics is thus a very worthwhile endeavor and should not be completely marginalized, a point which Quine fails to recognize.

On a different note, one could fear that Maddy is in fact opening a Pandora's box by granting proper mathematics this unique relationship with philosophy; for, what grants proper mathematics this right over that of proper-physics or proper-geology? While this concern is one I will later revisit, I do recognize there exist unique qualities of mathematics as compared to other scientific disciplines. Specifically, unlike the general practice of our scientific disciplines, mathematics is generally postulated outside of empirical research, and thus is not confined solely to scientific/empirical standards. Within the scientific disciplines, however, there is much reliance on one another. A chemist studying quantum theory will need physics to understand the actions of molecular motion, while a biologist will need chemistry to comprehend biological processes such as the sodium-potassium ATP pump.[34] Obviously, there exist independent approaches or points of emphasis which differentiate scientific disciplines; however, it seems that there exists a

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[34] This is the process (powered by the hydrolysis of ATP) by which a net positive charge is created outside of cells due to the continual, regulated cellular import of two positive potassium ions but export of three positive sodium ions from the cell.

hierarchical structure within the scientific disciplines. Such a structure is most likely not a clear linear progression of sub-theories, but clearly, as one progresses further in their field the more they rely on others for either reference or empirical verification.[35] Thus, there is no obvious exclusive "proper chemistry" or "proper biology;" however, examples in higher level physics seem to present separate issues. Maddy would dismiss these by claiming that higher level physics, which is apart of scientific recreation, eventually falls back on proper mathematics itself, thus any distinction between physics and extra-physics is just between science and meta-science (which includes proper mathematics). This makes sense, for if mathematics is indispensable in our science, then like anything else indispensable, we would always make some reference to it if we press hard enough. However, a clear distinction is not necessary here, and Maddy goes so far to argue that such distinction seems impossible.

At least initially, Maddy's naturalism seems a superior argument; however, its superiority is derived from its dismissal of Quine. If Maddy was unsuccessful in dismissing Quine but maintained her own naturalism, I would be hesitant to agree that Maddy's is the superior for the lack of commitments it grants us. Let us now attempt a defense of the indispensability argument. I will first discuss two ways which we may attempt to revive empirical holism in science. The first shall be an attempt to defend from the view of Quine, while the second shall be an addendum to our "best" theories. I will then discuss how we can reconcile the differences of scientific and mathematical goals, as well as the lack of any repercussion from mathematical practitioners by granting ontology through scientific naturalism.

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[35] This is by no means a proof of such a claim, but it is the general tendency.

XI(a). Indispensability: Defending Holism By Dismissing Scientists

The direction of my first argument is generally the most popular when attempting to rebut Maddy. Quine outlines qualities of our best scientific theories, which Maddy then use against him to demonstrate an example in which entities of a best theory were not granted ontological commitment by scientists in practice. Well then, clearly scientists were wrong! Molecular theory clearly was a best theory. Recall that philosophy is continuous with science, but the ideals of science are not necessarily scientific methodology. Scientists should have recognized this before asserting a right to grant or withhold ontological commitments, and thus, should have not attempted to wean in the atom by first making it a "useful fiction" and attempting to deny its existence. Clearly, novel theories like Cannizzaro's should be met with initial reservation; they should undergo several repeats of the experiments, as well as separate scientific experimentation for verification, but this is included in Quine's qualifications: "such testable consequences of the theory that have been tested have turned out well." Furthermore, the need for verification prior to accepting any novel theory is an integral part of overall science. Scientists made a mistake, therefore, when they tried to universally accept only one part of the theory when the whole theory was repeatedly tested and deserved complete acceptance. Well, as the continuity of philosophy and science works both ways, philosophers subscribed to Quine's scientific naturalism are justified in telling the scientist that they are being inconsistent in their practice of science and that such inconsistency is detrimental to their philosophical claim of addressing ontological and epistemological questions.

If we accept this rebuttal, then Maddy's main example against empirical holism falls apart; however, this is a scientific example about a purely scientific object – it does not mean we can ignore her inquiry into scientific empirical holism with regard to the mathematical objects to which scientists subscribe. She argues that Quine's best theories, when regarding mathematical objects, include uses of continuous mathematics, to which scientists knowingly do not grant ontological

commitment in our supposedly discrete world. Thus, she claims scientific holism in regard to mathematical entities is flawed.

Given the argument above, perhaps the same could be applied here: scientists are wrong. However, not wanting to push this argument too much, I argue that perhaps Maddy is confusing all useful mathematical entities of science with those that are truly indispensable to science itself. Quine does not draw specific distinctions when outlining his Indispensability argument, but Putnam, in his book *Philosophy of Logic*, fills in part of this gap by discussing the scientific necessity for predicative sets, compared to the scientific usefulness (but not necessarily necessity) of impredicative sets.[36] Thus, predicative sets are required for functionality and are indispensable in science, while impredicative sets are recommended for efficiency but not necessity required. Perhaps we can clarify Quine's unclear distinction by considering Quine's overall tendency to reduce the amount of ontological commitments we must make universally. Quine would not argue that we must have ontological commitments to impredicative sets just because they are used

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[36] A set is impredicative if its definition "defines an entity by means of a quantification over a domain of entities, including itself," while a predicative set has a definition which fails to do so. In simple terms, if a definition of set is a reference to how it compares to a larger set in which it is contained, then it is impredicative. The definition of natural numbers introduced at the beginning of this thesis is thus impredicative as it relies on the sets of numbers which precedes it. Other widely used definitions that map the natural numbers from the reals are also impredicative, as they are building the natural number sets by reference to a large set. The construction of mathematics from the impredicative definitions of its sets is the simplest way to understand the subject and has the simplest application when integrated into other scientific disciplines. However, while impredicative definitions carry a greater efficiency, they cannot be understood as the indispensable foundation of mathematics (and science) as they present some problems. First, they must eventually make reference to a predicative set, otherwise the references to other sets would be unqualified in application. Secondly, construction of mathematics based on impredicative definitions would leave open a strong argument against such a possibility with reference to Russell's Paradox. Predicative definitions of sets must therefore be our indispensable foundation. In a complex proof, Putnan shows that mathematics can be constructed solely through predicative definitions. A good example is the construction of pi: it is the ratio between a circle's circumference and its diameter. Putnam postulates that it is indeed possible (though complicated) to construct the set of rational numbers solely through reference to predicative sets, and thus he argues that the application of physics is possible, as physics relies heavily on the rational numbers (he uses the gravitational force equation between two objects as an example). Logic, too, he argues, can be constructed with some difficulty using only predicative sets. Both disciplines can be constructed much more efficiently through reference to impredicative sets, but this is not required.

frequently in our best theories: they are not indispensable to the overall theory as we can construct the same scientific theory with reference to predicative sets instead (but not the reverse). So perhaps Quine would argue that of course there are references to mathematical entities in our best scientific theories which are extremely useful, like continuous mathematics; however, they are not indispensable to our overall best theories and thus do not require ontological commitment. Maddy would want to argue that this throws out too much, but for Quine, the fewer ontological commitments the better. (See XI(d) for an address of this concern.)

XI(b). Indispensability: Defending Holism by Redefining "Best" Theories

A separate argument in defense of empirical holism would be to make an addendum to what constitutes our best scientific theories. Recall that Quine argued that molecular theory should be accepted as it had "simplicity, principle familiarity, scope, possibility for future growth, and continued verification of the theory." However, arguing that a scientific theory is one of our best when scientists cannot reach a consensus on the ontological consequences of such a theory seems to contradict the overall ideals of science, as science is rooted in a form of realism: our scientific theories are an attempt at an objective explanation of objects and their properties that we believe exist and exist independently of our mind. Note the slight difference here: I am not saying scientists or philosophers must argue over who is right (as in the previous section); rather, I am saying such an argument is not necessary, for clearly this theory should not be accepted as one of our best. This does not mean the theory is not extremely useful, nor does it mean we cannot apply it; rather, until the theory is *accepted by scientists as an honest representation of some part of reality*, it does not warrant ontological commitment. Note that I define reality here as the set of all existing objects, concrete and abstract, and their properties. Molecular theory would not pass this additional requirement

before the atom was accepted as an existing entity, as it failed to have a consensus of scientists on its ontological consequences.
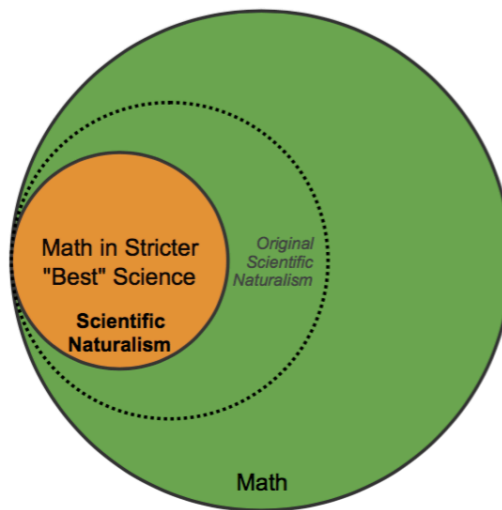
Quine's Naturalism
w/ Stricter Best

The benefits of this theory, which I argue make it stronger than the alternative "scientists are just incorrect," are that it clearly dispels any scientific theory that is false in reality. Furthermore, all scientific theories that use mathematics in a manner which is not built upon empirical evidence converging upon reality, do not constitute our best theories. This includes many uses of continuous mathematics in physics, as well as many scientific approximations and idealizations like ideal fluids (which is not based on converging evidence, as discussed in Section VIII(b)). Again, they are still useful theories! But for the purposes of which scientific theories we should consider when determining ontological commitments, they should not be included. The dismissal of Quine's defense of molecular theory comes with the benefit of a stronger argument for his scientific empirical holism overall, as it does not require us to explain the non-indispensability of many continuous mathematical entities. One impact of this alternative is that it does limit the amount of mathematical entities to which we eventually make ontological commitments; however,

I do not believe that Quine would have any issue with this result (nor does the mathematician, as I will later discuss).

A rebuttal from Maddy may be that this outline of what constitutes a best scientific theory is not uniform with what scientists in practice would say constitutes such a theory. To this I invoke my philosophical right to tell the scientists that they are wrong!

Finally, it should be noted that this new condition does grant a greater reference to practiced scientific methodology than present in Quine's original naturalism, as it appeals to only those theories which have a general consensus of scientists on their consequences. I agree with Maddy that Quine's argument is too ignorant of scientific practice; however, I do not embrace a redefinition of scientific naturalism that is built strictly from methodology. This form of naturalism – one based solely in practiced methodology – is one which even Popper dismissed, stating "methodology should not be taken for an empirical science."[37] While Popper does dismiss this strict form of naturalism, he does maintain that methodology remains an indispensable part of our science, which is why I maintain, unlike Quine, that we can never fully liberate ourselves from it. Maddy's naturalism, at least in regard to science, is not necessarily this extreme form of a redefinition, but I believe that her emphasis on the methodology of practitioners, rather than on the overall goals and ideals of science itself, is too high and is thus concerning.

Regardless of which argument holds best, it seems we have, at least initially, addressed some of the major concerns Maddy placed on empirical holism. I now need to address the differences in goals and ontological application which seem to separate mathematics and science overall.

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[37] Popper, *The Logic of Scientific Discovery*, 31.

XI(c). Indispensability: Reconciling Differences in Ontological Regard and Goals

Maddy argues that when mathematics is applied in science, even that which is non-idealized, there is no regard to the ontological commitment scientists are making concerning the mathematics they apply. The former two arguments can be applied here: Either, (i) when scientists fail to make ontological commitments regarding their best theories they are wrong or (ii) a respect for ontology is inherent in the "honest representation of some part of reality" which is needed to constitute a best theory so such theories are inconsequential. As pointed out by Mark Colyvan, unlike Maddy's claim, there exist several moments in science where a respect for ontology was granted.[38] Examples include initial concerns over the implications of calculus in science following Newton, as well as the scientific ontology of the imaginary number. Generally, it seems that "scientists do not worry too much about the ontological commitments of some mathematical theory, if that theory is already widely used," in contrast to novel mathematics applied to science. Thus, at some point, it seems that an analysis of mathematical ontology is usually undertaken by scientists. For theories where this is not the case, I would refer the argument back to (i) or (ii) mentioned above.

In regard to the apparent differences in goals between mathematics and science, I will not attempt to argue that such differences do not exist, but I claim that they are inconsequential when we look at both the practice of science and overall science itself. As I previously discussed in Section X, it seems that the further one advances in a particular scientific discipline, the more one references others. Though there exists some complex hierarchy of scientific disciplines, it seems that the further we push any science, we begin to make arguments in physics, which is then pushed into proper mathematics. However, even if chemistry is pushed into physics, we still recognize both

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[38] Colyvan, "In Defense of Indispensability," 51.

independently as they have separate goals themselves, yet we still do not hesitate in allowing them a place in our overall scientific naturalism. So while physics contains chemistry in the hierarchal structure of science, it maintains its independence. I argue this is no different than the expansion into math.

The biggest goal difference with which Maddy is concerned is that mathematics does not inherently concern itself with empirical inquiry whereas physics and other scientific disciplines do. Well, first of all, much of quantum physics and higher level theories lack this fundamental basis in empirical research, yet we still consider them important parts of our science. Maddy would push back at this point either by claiming that either these theories are just actually practices of proper mathematics or that they still stem from previous theories that have this empirical foundation. To the first, I would argue that Maddy is being unfaithful to the scientific practitioners who would most likely consider themselves both scientists and mathematicians.

Regardless, I would ask why should a loss of empirical concern between science and mathematics matter when there clearly is a subset of mathematics (physics) that is concerned with it and to which we limit our empirical inquiry? Like the move from chemistry to physics, we see the same impact of generalization: as we generalize our goals into the larger set, we loose specific goals unique to the former subset to embrace the generalized larger field. It just so happens that we loose the goal of empirical inquiry with our physics to mathematics jump, but (if we accept physics as the highest level of science) this just means that science always refers to mathematical claims, but not necessarily the reverse. We knew this all along, and Quine is not trying to explain an argument for the ontology of all mathematical objects, just the ones which are contained within

the subset of science. Expecting all of mathematics to have the same goals as its specific applications is unrealistic.[39]

XI(d). Indispensability: Ramifications of Mathematical Ontological Commitments

So far, I have addressed concerns regarding empirical holism, ontological recognition of mathematical objects by scientists, and the differences in the two subject's goals. The last concern I shall discuss is Maddy's claim that science has no right to commit us to mathematical entities without regard to mathematical methodology as a whole. To address this concern, I must again appeal to both the methodology and ideals of our best science and our best mathematics. If you accept my addendum to the qualifications for best scientific theories, then a respect for ontology is inherent in any theory which is an honest representation of some part of reality. Even if you don't accept this, or believe it to be unnecessary, it seems that the greatest difference between science and mathematics is this ontological respect.

Note here that science overall is a field which is very interested in both epistemological and ontological concerns, and science itself inherently makes ontological commitments to the objects perceived in the natural world. Furthermore, the practitioners of science are able to be uniform in their methodology for addressing ontological inquiries, as scientists collectively grant ontological commitments to many additional entities which are not immediately perceived, including the atom. Science and its practitioners also has the ability to grant ontological commitments to abstract objects if such an object can be determined through our empirical inquiry of the natural world. An example of this abstract entity would be the ideal plane, as empirical investigation is able to

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[39] This last point is similar to one also advocated by Colyvan; though, he sets up a different argument.

determine its emerging properties by investigation of the property convergence of concrete planes with decreasing amount of friction.

In practice, the mathematician lacks this uniform ontological methodology both concretely and abstractly (in regard to the necessities of practice), rather only having a uniform respect for mathematical epistemology through their proofs rooted in fundamental axioms. It is evident that Maddy agrees with this last point, given her conclusion that mathematical naturalism concludes in no ontological commitment due to a lack of ontological consensus among practicing mathematicians.

I argue that here Maddy, who is mainly concerned with mathematical practice, contradicts herself, for she argues that mathematicians don't need to agree on ontology to practice mathematics, while also arguing that allowing scientists to grant ontological commitments to mathematical entities would in fact restrict future mathematics.[40]

If this mathematical restriction is in regard to practiced mathematics, then clearly ontological commitments granted by scientists will have no impact, so her point is mute. Now, mathematics overall is certainly concerned with ontology and mathematics itself grants ontological commitments to at least some entities. But I see no possibility for future restriction as there are no specific commitments mathematics itself requires. So then why would we stop ourselves from asking an ontological question in a field that can methodologically address ontological questions (science) regarding an entity that plays a role in that field (mathematical objects), in order to address it solely in another field in which the entity play a role (general mathematics) that methodologically cannot address it and that overall ideals are mute on specific commitments? Maddy's justification consists of references to concerns which I have, at least initially, addressed above, and thus I argue

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[40] Maddy, *Naturalism in Mathematics*, 202.

we have good reason to no longer feel constricted by mathematical naturalism when addressing ontological questions within the philosophy of mathematics.

Lastly, it should be noted that all mathematics is important under its own right, and a vast majority of it is extremely useful in our science (including higher level theory that indirectly plays a role). I feel that Maddy places too much importance in the ontological commitments of science, which clearly are not necessary for mathematical or scientific usefulness (continuous mathematics/scientific approximations) or unproven application (high level set theory). We don't need to grant ontological commitment to all or any mathematical entities to agree on that importance, so limiting our commitments to those made solely through scientific inquiry will not deter from this understanding.

Overall, this dissection of Maddy's argument is by no means a complete proof and rests on some premises which philosophers of science or mathematics may quickly attack, but it does highlight some possible weaknesses in Maddy's argument which should be either addressed or clarified.

XII. Maddy's Set Realism

Assuming for a moment that I have successfully demonstrated unamendable issues of an argument on which Maddy has based three books, let us examine an alternative argument from Maddy that rests on the premise that scientific naturalism and indispensability grants the existence of sets. The argument – that sets exist concretely – is outlined in Maddy's 1990 book *Realism in Mathematics* and is a specific argument within her view of set theoretic realism. While she eventually dismissed set theoretic realism in her 1997 book *Naturalism in Mathematics* for its failure to disprove $V = L$ and it dependence on the realism prescribed by Quine's scientific naturalist, she never specifically dismissed the argument within this theory, even going so far to state that a modification

of it may still be applicable to philosophical questioning within her mathematical naturalism.[41] As Maddy is unconcerned with elaborating on ontological questions within the philosophy of mathematics, she never attempts such a modification. I will not attempt such a modification either; rather, I will assume from Quine's indispensability that sets exist in some way and will show that even by granting such a premise, Maddy's argument still presents areas of concern.

Maddy's argument for the concrete existence of sets is only possible through an alteration of Benacerraf's dilemma, which she makes by arguing that his underlying premises are based on outdated epistemology.[42] Maddy claims that there are two types of knowledge: inferential knowledge, which is obtained by logical deduction from theories; and non-inferential knowledge, which is caused by perceptual experience of an object itself. Note, however, that it is necessary to distinguish non-inferential knowledge from immediate sensory experience. As causal connections through immediate sensory experience can be flawed in certain environments and mental states, or flawed through the fact that we have not yet learned to perceive different objects within our senses, we can not purely rely on them in determining the knowledge we have of the world. Thus, our knowledge is not necessarily based on causal connections as Benacerraf argued, but rather on the overall reliability of the argument. As a result, Benacerraf's restriction to causal theory is misguided and overall philosophical theory of mathematics must present an account of knowledge purely through either an inferential process or through a mechanism by which such a belief is immediately granted non-inferentially.

From Quine, we have a good account of mathematical truth regarding sets, for we are confident in their existence, but we must reassess our account of knowledge. To understand how we come to have knowledge of sets, we should first attempt to form a line of causal references which

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[41] Maddy, An Interview with Penelope Maddy.

[42] Maddy, "Mathematical Epistemology: What Is the Question?"

leads us back to the initial dedication of "sets." We shouldn't feel restricted to this causal argument, but it's certainly a reliable place to start. This initial debut can take two forms, either by description or by ostension. As any description would entail a reference to itself (e.g. sets are a kind of which the <u>set</u> of {books, numbers, etc.} is a sample), it must be dismissed. Therefore, this initial dedication was by ostension, in which the dedicator picks out samples of a kind, with "set" referring to the kind of which these samples are members. Note that we can therefore refer to sets with little previous knowledge needed; however, as determined earlier, we must be able to justify this with an inferential or non-inferential process.

Consider our perception of any concrete object. Our knowledge of this object is non-inferential, as we learn to immediately perceive the object after we have built enough neural connections in our brain (i.e. "object detectors") to unconsciously make such a distinction. This learned perception provides us our knowledge of any concrete objects.

Maddy argues that how we perceive sets is exactly the same.[43] As a child, we learn to differentiate sets of different sizes, which eventually leads to perceptual (non-inferential) knowledge of sets of small numbers. Note that Maddy claims that when we see any small collection of objects, the number we associate with that collection is in fact just a property associated with that set of objects. This is only for small sets, as empirical evidence testing human reaction times suggests that we perceive small numbers, rather than count them (which would be an inferential process). It is important to note that Maddy claims we associate number properties not with the elements within the set, but with the set itself as its size, as the set's elements can be broken down into greater quantified entities such as atoms.[44] Thus, according to Maddy, we have an acceptable argument for the concrete existence of small sets: we have an account of truth through Quine and we have

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[43] Maddy, *Realism in Mathematics*, 50–67.
[44] See Irvine, "Frege on Number Properties." for further justification that number is a property.

an account of knowledge through our non-inferential understanding of numbers and sets, which is justified scientifically by the building of neuronal connection as we learn to perceive sensory experience.

Maddy's justification that we are perceiving sets, as compared to perceiving entities such as aggregates, concepts, or classes, is based on two premises, the first of which being that as we already know sets exist (by Quine), our perception of an entity which fits all their properties must just a concrete example. The second premise comes from the realization that because our non-inferential knowledge is very limited, the objects we perceive must be as equally simple. Essentially, when we perceive three eggs, all we know is that we are perceiving *something* with the number property of three; all other knowledge built from this – atoms, arrangement, designations – is theoretical (inferential knowledge). As aggregates and classes both depend on some degree of relatable properties between its elements, processing our sensory experience of them would require some form of inferential analysis. Furthermore, these entities present the possibility for paradoxes, such as Russell's.

If Maddy's realism argument for concrete sets holds, then it may provide some advantages over a purely Quinean holistic argument (i.e. the concrete existence of some sets is an indispensable part of science, and thus we are happily committed to such existence). This advantage would come from her argument's center on the obviousness of mathematics in the physical world, rather than Quine's center on concrete sets being hidden in scientific theory.[45]

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[45] Maddy also applied this theory to our inferential knowledge of higher level sets: We have non-inferential knowledge of small numbers and sets, as well as how they combine to form other small numbers (i.e., the axioms of pairing and union). We then build upon this knowledge inferentially with theories that are not directly verifiable but depend on the explanation and advancement of this lower theory.

XIII. Weaknesses of Maddy's Set Realism

On first approach, Maddy's set theoretic realism, which as it is based in physiological and empirical science, seems a good candidate as an appropriate answer to the philosophical question "Do small sets exist concretely?" As Maddy's argument is based heavily in the link between perception and reality, we must restrain the vast floodgates of philosophical arguments by limiting our analysis to the confines of a naturalism that maintains that our empirical science is the only available method for epistemology. Maddy argues that we originally come to know sets by ostension, which is justified by our scientific knowledge of the construction of axioms in certain areas of our brain. Once we learn this distinction, we can have perceptual acknowledgment of this existence of some small sets just like we have perception of anything else. I want to show that unlike Maddy's claim, her "perception" of sets is unlike our perception of any other object.

In *Realism in Mathematics*, Maddy argues that our perception of sets in not unlike our perception of triangles, only just much more complicated. Specifically, she states our perceptions of a triangle is a result of formed neuronal assemblies that "respond to the apex of any similar triangle."[46] Furthermore, she states, "The cell assembly is what permits the subject to see a triangle with identity. The trick is to see a series of patterns as constituting views of a single thing…the ability to see triangles develops over time through…seeking out corners and comparisons."[47] Throughout her argument, including her references to perceptions of sets, she is uniform in this link between sensory experience (e.g. sight) and perception. This is not a radical claim: to have perception of an object, one must have sensory experience of that object. My general concern, however, is that unlike Maddy's claim, our sensory experience of sets seems to be defined by the concrete elements of the set, not the set itself. Consider the set of four books. To be uniform with

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[46] Maddy, *Realism in Mathematics*, 56.
[47] Ibid., 57.

how we perceive any other object, we must actually see the set. Well, clearly, I see four books grouped together, but do I see a set, or is this grouping only a result of natural collections or relations we make in our mind?

Very basically, sight is caused by the light radiating off a physical object, hitting our eyes, activating a signal to our brain, and then activating the appropriate neurons. But the light never hits the "set," it hits the four books. More specifically, light is refracted off the millions of atoms which compose the books themselves. Sensory experience and perception of the individual elements of the set do not explain the jump from perception of those individual objects to perception of the overall set, as there does not exist any additional concrete property that can provide such a differentiation. This argument works for all forms of senses – we can hear two birds singing and realize a set of birds; however, the sound waves which eventually translate into neuronal stimuli are caused by the birds themselves, not a set of the birds.

Maddy's pushback would be to work backwards: the number 4, which is perceptual, cannot be associated with these individual elements, as these elements can be broken down (maybe indefinitely), thus it must be associated with the size of a set, which we must also perceive. Because perception is derived from sensory experience, we must be seeing a set.

I do not disagree that small numbers, such as four, are perceptual, nor do I disagree that we attribute such a number to the size of a set of objects. My issue is that this form of "perception" about a mathematical object and a mathematical property is clearly different from our perception of any other object. One could say we have *direct* sensory experience of objects other than sets, while Maddy has made a good case for the *indirect* sensory experience of sets. Analogously, with other objects we can explain, with our scientific reasoning, the path from sensory experience to perception (i.e. sensory experience -> perception), but we also have a recognition that our perceptions must be from sensed objects (i.e. perception -> sensory experience); thus, overall, we

are reassured by our science in our belief that our perception is from the sensory experience of concrete objects (perception ⇔ sensory experience). We lack, however, this reassurance of scientific reasoning when we solely assume that our perception of sets is based on sensory experience (perception -> sensory experience). Maddy claims we can be assured in our perception of sets just as much as our perception of anything else; clearly this is not the case.

Furthermore, as Maddy points out, one person could perceive a set of two sets of two books, while another could perceive just one set of four books. How can we explain this? Well, this unique differentiation can be caused by distance, property differentiation, and brute force. Distance and property differentiation can be explained by individual sensory experiences concerning the elements of the set(s); however, again, this is related to the sensory properties of the individual elements of the set, not the overall set itself. The latter, "brute force" method, is when we force ourselves to perceive individual sets within a set of objects. Arguably, this is similar to "seeing" a book, but perceiving "a cover" and "344 pages." However, once again, when applying this method to set differentiation, we see that any differentiation is not based on any sensory experience. It is only made through different groupings we make in our mind; thus this is another example of perception without sensory justification.

Now that we have shown our perceptions of sets to be a weaker form of knowledge than our perception of any other object. Let's discuss how this affects Maddy's argument considering her definition of non-inferential knowledge. Defenders of Maddy's unjustified perception argue that the lack of any sensation of sets is inconsequential when we consider that perception, not necessarily sensation, is needed to justify non-inferential knowledge. For example, Staughan Lavine claims that such a perception of sets is like "see[ing] the shadow of someone carrying a knife, you acquire the perceptual belief that there is a mugger behind you [without seeing] the

mugger."[48] To Lavine's benefit, recall that in Maddy's original justification she does in fact reference a distinction between immediate sensory experience and perception, claiming that causal connections with immediate sensory experience is not how we should justify our non-inferential knowledge, rather it should be through our perceptions, which have been developed over time. However, as we cannot account for this development, we could perhaps make an intial argument claiming that sets don't constitute this non-inferential knowledge. Furthermore, there is a clear difference between perceiving a mugger from his shadow and perceiving a set from its elements. Our perceptual belief of the mugger is made because we have learned that a shadow is directly linked to an object. This "learning" (i.e. neural connection building) occurred because over time we associated a shadow with a person/object to which we immediately turned around and physically sensed. With sets, rather, there seems to be a lack of a similar learned justification, so I am hesitant to accept Lavine's analogy.

Another issue concerning the justification of non-inferential knowledge comes from reference to Maddy's addendum to Benacerraf that the differentiation and application of knowledge is based not on casual connections but on the reliability of the argument.[49] To say that we perceive sets without immediate sensation may be completely acceptable; however, I am skeptical to accept this as non-inferential knowledge due to the seeming lack of "strong reliability" from our scientific knowledge that is granted to (seemingly) every other object. Perhaps the inference from the sensation/perception of every other object onto sets actually is suitability reliable, but I feel this is a weak argument.

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[48] Lavine, "Review of Realism in Mathematics," 323. Note that Lavine eventually dismisses Maddy's set realsim, but with a different argument.

[49] Maddy, "Mathematical Epistemology: What Is the Question?," 49.

On a separate note, let's assume that Maddy did explain a modification of this argument that fit within her dismissal of Quine and proposition of mathematical naturalism, and that she was able to explain through it that sets exist in some way. Well, I have argued that there exists a distinction between seeing a set versus seeing any other object. I further argue that such a distinction arises because we are attempting to apply a mathematical definition of "set" onto our scientific understanding of sensation and perception. Given that Maddy's naturalism is built from the separation of scientific and mathematical methodologies, it seems we cannot apply our scientific understanding of perception to the perception of a set, as the difference I exposed highlights a distinction between scientific and mathematical perception. This distinction may force us into an argument that requires appealing to all of mathematics, leaving behind a solely empirical argument.

XIV. General Analysis & Future Applications

If Quine's indispensability holds and we accept the modification of what constitutes our best scientific theories, then the corresponding mathematical objects, which certainly include at least some sets, do in fact exist. The specific nature of this existence is up for debate, but there certainly exists the potential for the holistic analysis of our best theories that returns ontological commitment to both concrete and abstract sets.

If Maddy's arguments against Quine are not properly addressed herein, then we only have ontological commitments to scientific entities of which a consensus of scientists agrees exist, and we can have no ontological commitments to any mathematical object.

Possibilities for future study include application of Maddy's argument for the concrete existence of sets to her mathematical naturalism while bypassing or rebutting my concerns. Additionally, to define a set of our best scientific theories and list the corresponding mathematical

entities that, by Quine, we therefore can commit existence. Finally, to explore what else is indispensable to science and attempt to expose any issues or contradictions that may arise by ontological commitments to these entities.

XV. Conclusion

In this thesis I have outlined two contemporary naturalistic arguments for how we can have ontological commitments to some mathematical entities. I have discussed how both Quine's and Maddy's forms of naturalism successfully overcome Benacerraf's dilemma and how they present theories that provide plausible accounts of truth and knowledge. I have argued that both theories overcome the general concerns of naturalistic arguments, as well as explained the justification for Quine's use of empirical holism that is central to his naturalism. Furthermore, I have outlined and responded to Maddy's argument against Quine's empirical holism in science, as well as her argument that the ontology of mathematics does not concern the scientist, by providing two options: uphold the right of philosophers to tell scientists when their methodology is incorrect, or make an addendum to Quine's definition of what constitutes our best scientific theories. Additionally, I have addressed Maddy's concerns regarding the difference between mathematical and scientific goals by arguing that we should not be surprised or concerned that we lose specific goals as we generalize into larger theories that encompass the smaller, more specific theories (such as mathematics encompassing physics). Furthermore, I have argued that if we assume that our defense of indispensability is valid and sets do exist, then Maddy's argument that sets exist concretely has several areas of concern as it fails to show that the knowledge of our perception of sets is just as strong as the knowledge of our perception of any other object.

My arguments against Maddy and my critiques of Quine are an attempt to highlight general weaknesses present in their views. These concerns should be additionally elaborated upon,

critiqued, or dismissed, as it is impossible to properly address these issues without writing at least a small book. Hopefully, my arguments provide enough of an introduction and overview of how these more elaborate arguments may be presented and what additional questions should be addressed.

XVI. Bibliography

Aristotle. "Metaphysics." Translated by W. D. Ross. The Internet Classics Archive, 350AD. http://classics.mit.edu/Aristotle/metaphysics.5.v.html.

Benacerraf, Paul. "Mathematical Truth." In *The Philosophy of Mathematics*, edited by W. D. Hart, 14–30. New York: Oxford University Press, 1996.

Bernays, Paul. "On Platonism in Mathematics." In *Philosophy of Mathematics: Selected Readings*, edited by Paul Benacerraf and Hilary Putnam, 2nd ed., 258–71. New York: Cambridge University Press, 1983.

Best, Myron G. *Igneous and Metamorphic Petrology*. John Wiley & Sons, 2013.

Carnap, Rudolf, and W. D. Hart. "Empiricism, Semantics, and Ontology." In *Philosophy of Mathematics*, 241–57. New York: Oxford University Press, 1996.

———. Meaning and Necessity - A Study in Semantics and Modal Logic. Read Books Ltd, 2013.

———. *The Logical Syntax of Language*. Translated by Amethe Smeaton. 3rd ed. New York: The Humanities Press, 1951.

Colyvan, Mark. "In Defence of Indispensability." Philosophia Mathematica 6, no. 1 (1998): 39–62.

Euclid, and Thomas Little Heath. *The Thirteen Books of Euclid's Elements*. 2d ed., rev. with additions. New York: Dover Publications, 1956.

Fleissner, William. "Normal Moore Spaces in the Constructible Universe." Proceedings of the American Mathematical Society 46, no. 2 (1974): 294–98. doi:10.2307/2039914.

Forrest, Peter. "Is Space-Time Discrete or Continuous? An Empirical Question." Synthese 103, no. 3 (1995): 327–54.

Frege, Gottlob. "The Concept of Number." In *Philosophy of Mathematics: Selected Readings*, edited by Paul Benacerraf and Hilary Putnam, 2nd ed., 130–59. New York: Cambridge University Press, 1983.

Gregory, Paul. "Set Theory [Forthcoming]." In Formal Logic. Peterborough, ON: Broadview Press, 2016.

Irvine, Andrew D. "Frege on Number Properties." Studia Logica: An International Journal for Symbolic Logic 96, no. 2 (2010): 239–60.

Kunen, Kenneth. *The Foundations of Mathematics. Studies in Logic: Mathematical Logic and Foundations 19*. London: College Publications, 2009.

Lavine, Shaughan. "Finite Mathematics." Synthese 103, no. 3 (1995): 389–420.

———. Review of Review of Realism in Mathematics, by Penelope Maddy. The Journal of Philosophy 89, no. 6 (1992): 321–26. doi:10.2307/2026715.

Liggins, David. "Quine, Putnam, and the 'Quine-Putnam' Indispensability Argument." Erkenntnis (1975-) 68, no. 1 (2008): 113–27.

Maddy, Penelope. An Interview with Penelope Maddy. Interview by The Dualist. Dictated, March 2009. http://www.socsci.uci.edu/~pjmaddy/bio/DualistVol15_Maddy.pdf.

———. "Mathematical Epistemology: What Is the Question?" The Monist 67, no. 1 (1984): 46–55.

———. *Naturalism in Mathematics*. New York: Oxford University Press, 1997.

———. "Perception and Mathematical Intuition." In *The Philosophy of Mathematics*, edited by W. D. Hart, 114–41. New York: Oxford University Press, 1996.

———. *Realism in Mathematics*. New York: Oxford University Press, 1990.

Mill, John Stuart. *A System of Logic, Ratiocinative and Inductive, Being a Connected View of the Principles of Evidence, and the Methods of Scientific Investigation*. J. W. Parker, 1851.

Popper, Karl. "Science as Falsification." In *Conjectures and Refutations: The Growth of Scientific Knowledge*, 33–39. London: Routledge and Keagan Paul, 1963.

———. *The Logic of Scientific Discovery*. 6th ed. New York: Routledge Classics, 2002.

Putnam, Hilary. "Mathematics without Foundations." In *The Philosophy of Mathematics*, edited by W. D. Hart, 168–84. New York: Oxford University Press, 1996.

———. *Philosophy of Logic*. London: Allen and Unwin, 1972.

———. "What Is Mathematical Truth?" Historia Mathematica 2, no. 4 (November 1, 1975): 529–33. doi:10.1016/0315-0860(75)90116-0.

Quine, W. V. "Naturalism: Or, Living Within One's Means." Dialectica, no. 49 (1995): 251–61.

———. "On What There Is." In *From a Logical Point of View*. Cambridge, MA: Harvard University Press, 1953.

———. "Posits and Reality." In *Ways of Paradox and Other Essays*, 233–41. Cambridge, MA: Harvard University Press, 1976.

———. "Truth By Convention." In *Philosophy of Mathematics: Selected Readings*, edited by Paul Benacerraf and Hilary Putnam, 2nd ed. New York: Cambridge University Press, 1983.

———. "Two Dogmas of Empiricism." In *The Philosophy of Mathematics*, edited by W. D. Hart, 31–51. New York: Oxford University Press, 1996.

———. *Word and Object*. Cambridge, MA: MIT Press, 1960.

Tait, W. W. "Truth and Proof: The Platonism of Mathematics." In *The Philosophy of Mathematics*, edited by W. D. Hart, 142–67. New York: Oxford University Press, 1996.

Tall, Franklin D. "How Separable Is a Space? That Depends on Your Set Theory." Proceedings of the American Mathematical Society 46, no. 2 (1974): 310–14. doi:10.2307/2039917.

Wedberg, Anders. *Plato's Philosophy of Mathematics*. Stockholm: Almqvist & Wiksell, 1995.

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