Prob (2) page 111 Byerly's Int Cal

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$$ pass to polar co-ords

$$r^2 = c^2\,\frac{b^4\cos^2\varphi + a^4\sin^2\varphi}{(b^2\cos^2\varphi + a^2\sin^2\varphi)^2}$$

$$A=\tfrac12\int r^2\,d\varphi = \tfrac12 c^2\int\frac{b^4\cos^2\varphi + a^4\sin^2\varphi\, d\varphi}{(b^2\cos^2\varphi + a^2\sin^2\varphi)^2}$$

$$=\tfrac12 c^2\int\frac{b^4\cos^2\varphi\, d\varphi}{(b^2\cos^2\varphi+a^2\sin^2\varphi)^2}+\tfrac12 c^2\int\frac{a^4\sin^2\varphi\, d\varphi}{b^2\cos^2\varphi+a^2\sin^2\varphi}$$

$$=\tfrac12 c^2\int\frac{\sec^2\varphi\, d\varphi}{\left(1+\frac{a^2}{b^2}\tan^2\varphi\right)^2}+\tfrac12 c^2\int\frac{\operatorname{cosec}^2\varphi\, d\varphi}{\left(1+\frac{b^2}{a^2}\cot^2\varphi\right)^2}$$

(1) (2)

To integrate (1) place $z=\frac{a}{b}\tan\varphi$.

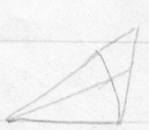
" " (2) " $z'=\frac{b}{a}\cot\varphi$

$$A=\tfrac12 c^2\int\frac{\frac{b}{a}\,dz}{(1+z^2)^2}-\tfrac12 c^2\int\frac{-\operatorname{cosec}^2\varphi\, d\varphi}{\left(1+\frac{b^2}{a^2}\cot^2\varphi\right)^2}$$

$$=\tfrac12 c^2\int\frac{\frac{b}{a}\,dz}{(1+z^2)^2}-\tfrac12 c^2\int\frac{\frac{a}{b}\,dz'}{(1+z'^2)^2}$$

$$=\tfrac12 c^2\frac{b}{a}\int\frac{dz}{(1+z^2)^2}-\tfrac12 c^2\frac{a}{b}\int\frac{dz'}{(1+z'^2)^2}$$ Now by reduction formula

$$=\tfrac12 c^2\frac{b}{a}\,\tfrac12\int\frac{dz}{1+z^2}-\tfrac12 c^2\frac{a}{b}\,\tfrac12\int\frac{dz'}{1+z'^2}$$



$$=\tfrac14 c^2\frac{b}{a}\tan^{-1}z+\tfrac14 c^2\frac{a}{b}\cot^{-1}z'$$

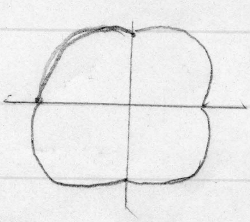
$$=\tfrac14 c^2\frac{b}{a}\tan^{-1}\left(\frac{a}{b}\tan\varphi\right)+\tfrac14 c^2\frac{a}{b}\cot^{-1}\left(\frac{b}{a}\cot\varphi\right)$$

$$=\tfrac14 c^2\frac{b}{a}\varphi+\tfrac14 c^2\frac{a}{b}\varphi$$

Between the limits $\varphi=0$ & $\varphi=\frac{\pi}{2}$

$$A=\tfrac14 c^2\frac{\pi}{2}\left(\frac{b}{a}+\frac{a}{b}\right)=\tfrac14\,\frac{\pi c^2}{2ab}(a^2+b^2)$$

$$\&\ 4A=\frac{\pi c^2}{2ab}(a^2+b^2)$$



$m = 1$
$p = -2$
$a = 1$
$b = 1$
$n = 2$

$\therefore \frac{z(1+z^2)^{-1}}{1.2(-1)} = \frac{z}{2(1+z^2)} + \frac{1}{2}\int \frac{\partial z}{1+z^2}$

$\frac{a^2}{2} \Big] \frac{\pi}{8}$

$x = 0$

$\varphi = -\frac{\cos\varphi}{\sin\varphi} = -\cot\varphi$

$y = 0$

$\varphi = \frac{\sin\varphi}{\cos\varphi} = \tan\varphi$

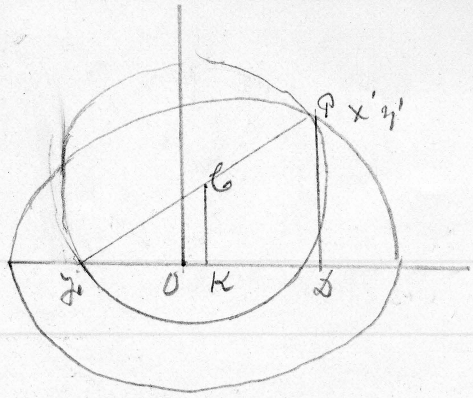
$\frac{a^2}{2}\int \cos 2\varphi \sin 2\varphi \, d2\varphi$

$= \frac{a^2}{2} \frac{\sin^2 2\varphi}{2}$

$\frac{1}{4}c^2 \frac{b^2}{a^2} \cdot \frac{a}{b} \tan^{-1} \frac{a}{b}\tan\varphi$

$\frac{1}{4}c^2 \frac{b}{a} \tan^{-1} z$

$z = \frac{a}{b}\tan\varphi$



$FD = c + x'$

$FK = \frac{c+x'}{2}$

$OK = \frac{c+x'}{2} - c = \frac{x'-c}{2}$

$GK = \frac{y'}{2}$

$FP = a + \varepsilon x'$

$GP = \frac{a+\varepsilon x'}{2}$

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$$

Equation of circle $\left(x - \frac{x'-c}{2}\right)^2 + \left(y - \frac{y'}{2}\right)^2 = \left(\frac{a+\varepsilon x'}{2}\right)^2$

$$(2x - x' + c)^2 + (2y - y')^2 = (a+\varepsilon x')^2 \qquad (1)$$

$$-2(2x - x + c) - 2(2y - y')\frac{\partial y'}{\partial x'} = 2\varepsilon(a + \varepsilon x')$$

$$\frac{\partial y'}{\partial x'} = \frac{b^2 x'}{a^2 y'}$$

$$-2x + x' - c + \frac{2b^2 y x'}{a^2 y'} - \frac{b^2 x'}{a^2} = \varepsilon a + \varepsilon^2 x'$$

$$-2x + \frac{a^2 - b^2}{a^2}x' - c + \frac{2yb^2x'}{a^2y'} = \varepsilon a + \cancel{\varepsilon^2 x'}$$

$$-2x + \frac{2b^2 y x'}{a^2 y'} = 2c$$

$$-x + \frac{b^2 y x'}{a^2 y'} = c$$

$$y' = \frac{b^2 y x'}{a^2(c+x)}, \quad \frac{y'^2}{b^2} = \frac{b^2 y^2 x'^2}{a^4(c+x)^2}$$

$$\frac{x'^2}{a^2} = 1 - \frac{b^2 y^2 x'^2}{a^4(c+x)^2}$$

$$x'^2 = a^2 - \frac{b^2 y^2 x'^2}{a^2(c+x)^2}$$

$$x'^2 = \frac{a^4(c+x)^2}{a^2(c+x)^2 + b^2y^2}$$

$$x' = \frac{a^2(c+x)}{\sqrt{a^2(c+x)^2 + b^2y^2}} \quad \& \quad y' = \frac{b^2 y}{\sqrt{a^2(c+x)^2 + b^2y^2}}$$

Substitute the values of x' & y' in (1)

$$\left(2x + c - \frac{a^2(c+x)}{\sqrt{a^2(c+x)^2 + b^2y^2}}\right)^2 + \left(2y - \frac{b^2y}{\sqrt{a^2(c+x)^2 + b^2y^2}}\right)^2 = a + \frac{\varepsilon a^2(c+x)}{\sqrt{a^2(c+x)^2 + b^2y^2}}$$

This equation reduces to $x^2 + y^2 = a^2$

Make the reduction.

$$c\eta^2 + 2bx\eta = 1 - ax^2$$

$$\eta^2 + \frac{2b}{c}x\eta = \frac{1-ax^2}{c}$$

$$\eta = -\frac{bx}{c} \pm \sqrt{\frac{1-ax^2}{c} + \frac{b^2x^2}{c^2}}$$

$$\eta = -\frac{bx}{c} \pm \frac{1}{c}\sqrt{c - (ac - b^2)x^2}$$