The $\lambda-$property and Isometries of the Higher Order Schreier Spaces

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ABSTRACT

For each $n \in \mathbb{N}$, let $\mathcal{S}_n$ be the Schreier set of order n and $X_{\mathcal{S}_n}$ be the corresponding Schreier space of order n. In their 1989 paper *The $\lambda-$property in Schreier space S and the Lorentz space $d(a,1)$*, Th. Shura and D. Trautman proved that the Schreier space of order 1 has the $\lambda-$property. This thesis extends the theorem by proving the $\lambda-$property for the Schreier spaces of any order and the uniform $\lambda-$property (stronger than the $\lambda-$property) for the $p-$convexification of these spaces. Furthermore, using what we know about extreme points of the unit balls, we are able to characterize all surjective linear isometries of these spaces.

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CONTENTS

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INTRODUCTION

In 1930, Schreier constructed the Schreier space $X_{\mathcal{S}_1}$ in [12] as a counterexample to a question of Banach and Saks. The standard basis of $X_{\mathcal{S}_1}$ has the property that it is weakly null, but there is no subsequence that *Cēsaro* sums to 0. Since the norm of the Tsirelson space is closedly related to the norm of the Schreier space - the first example of a Banach space in which neither an l_p space nor a c_0 space can be embedded, the Schreier space has been studied extensively in [7]. In particular, the space is hereditarily$-c_0$, meaning that every closed infinite dimensional subspace has a sequence of vectors equivalent to the unit vector basis of c_0. Consequently, ℓ_1 does not embed in $X_{\mathcal{S}_1}$. As the definition of $X_{\mathcal{S}_1}$ depends on $\mathcal{S}_1$, a collection of finite subsets of $\mathbb{N}$, we will also consider $X_{\mathcal{F}}$ for other families of finite subsets of $\mathbb{N}$. Specifically, for each $n \in \mathbb{N}$, there is a collection $\mathcal{S}_n$ of finite subsets of $\mathbb{N}$ with greater complexity. The objective of this thesis is to investigate several geometric properties of the Schreier space, its higher order spaces, and their p-convexification.

In [3], R. Aron and R. Lohman introduced geometric properties for Banach spaces, called the $\lambda-$property and uniform $\lambda-$property. Since then, the $\lambda-$property has been extensively studied by many authors over the past 25 years. In 1989, Th. Shura and D. Trautman proved, in [14], that the Schreier space has the $\lambda-$property and the set of extreme points is countably infinite. In [6], K. Beanland, N. Duncan, M. Holt, and J. Quigley proved several results for combinatorial Banach spaces and, in particular, showed that the set of extreme points the unit ball of $X_{\mathcal{F}}$ is at most countable for every regular family $\mathcal{F}$. This thesis builds on these works and proves the $\lambda-$property for the Schreier space of any order and the uniform $\lambda-$property for the $p-$convexification of these spaces.

Next, the thesis characterizes isometries of Schreier spaces. Given a Banach space X, we denote by $\operatorname{Isom}(X)$ the group formed by all surjective linear isometries of X. The characterization of the isometries plays a central role in the field of geometry of Banach spaces and can be found already in the famous Banach's treatise of 1932 [5], in which he gives the general form of isometries of classical spaces, such as c, c_0, $C(K)$, ℓ_p and $L_p, 1 \leq p < \infty$. Characterizations for other spaces can be found in [9]. In this thesis, we use results concerning extreme points of Schreier spaces to exhibit the general form of the elements of $\mathrm{Isom}(X_{\mathcal{S}_n})$, with $n \in \mathbb{N}$.

The thesis is structured as follows. Chapter 2 introduces the reader to general concepts of Banach spaces, the $\lambda-$property and surjective linear isometries. Chapters 3 and 4 in Part II introduce the Schreier space, prove its $\lambda-$property, and characterize all of its isometries. All of these results are generalized to the higher order spaces in Chapters 6 and 7. Though the results in Part II are implied by the results in Part III, we decide to devote Part II solely to the Schreier space for two reasons. First, our proof of the $\lambda-$property for the Schreier space clarifies several points implicitly made in the proof of Th. Shura and D. Trautman. Second, understanding our proof of the Schreier space case makes it much easier to understand the proof in the general case. The last part includes an interesting result from the way we define the Schreier sets; that is, we can find Fibonacci sequences of any order by counting a family of generalized Schreier sets in a certain way.

Part I

PRELIMINARIES

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PRELIMINARIES

2.1 BANACH SPACES

We begin with the definition of a Banach space. We focus our attention on Banach spaces of real scalars, though our definitions holds for complex scalars.

Definition 1. *Suppose X is a real vector space. A norm $\|\cdot\|$ is a real-valued function satisfying the following three conditions:*

1. $\|x\| \geqslant 0$ *for all* $x \in X$, *and* $\|x\| = 0$ *if and only if* $x = \vec{0}$;
2. $\|\lambda x\| = |\lambda|\|x\|$ *for all* $x \in X$, $\lambda \in \mathbb{R}$;
3. $\|x+y\| \leqslant \|x\| + \|y\|$ *for all* $x, y \in X$.

The pair $(X, \|\cdot\|)$, that is, the linear space X equipped with the norm $\|\cdot\|$, is called a normed linear space.
A normed linear space $(X, \|\cdot\|)$ is complete provided all Cauchy sequences in X have limits in X.

Definition 2. *The normed linear space X is a Banach space provided X is complete with respect to its norm.*

Let $\mathbb{R}^\infty$ denote the vector space consisting of all sequences of real numbers. All Banach spaces we consider will be subspaces of $\mathbb{R}^\infty$. Each vector of these spaces is an infinite sequence of real numbers.

Definition 3. *Given an incomplete normed linear subspace X of $\mathbb{R}^\infty$, there exists a unique normed linear space $\hat{X}$, also a subspace of $\mathbb{R}^\infty$, such that X is a subspace of $\hat{X}$, $\hat{X}$ is complete with respect to X's norm, and X is dense in $\hat{X}$. The space $\hat{X}$ is a Banach space and is called the* completion *of X.*

Let X be a Banach space. The unit ball of X, denoted $B(X)$, is $\{x \in X : ||x|| \leq 1\}$, and the unit sphere of X, denoted $S(X)$, is $\{x \in X : ||x|| = 1\}$.

Definition 4. *A vector $x \in B(X)$ is an extreme point if $x = \lambda y + (1-\lambda)z$, where $0 < \lambda < 1$ and $y, z \in B(X)$ implies $x = y = z$. Equivalently, $x = \frac{1}{2}(y+z)$, where $y, z \in B(X)$ implies $x = y = z$.*

Let $E(X)$ be the set of all extreme points in the unit ball of X. Note $||x|| = 1$ for all $x \in E(X)$.

Definition 5. *Let $x = (x(1), x(2), x(3), \ldots)$ be a vector in a Banach space X. The support of x is denoted* $\operatorname{supp} x = \{i : x_i \neq 0\}$, *and* $\max \operatorname{supp} x$ *is the maximum element in* $\operatorname{supp} x$ *(if it exists).*

Now we provide a couple examples of Banach spaces. The two most basic Banach spaces are the ℓ_p and the c_0.

Definition 6. *For $1 \leqslant p < \infty$, $(l_p, \|\cdot\|_p)$ is a Banach space, where*

$$l_p = \left\{ (a_i)_{i=1}^{\infty} : \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{\frac{1}{p}} < \infty \right\}$$

and $\|(a_i)\|_p = (\sum_{i=1}^{\infty} |a_i|^p)^{\frac{1}{p}}$.

Definition 7. *As another example, $(c_0, \|\cdot\|_0)$ is a Banach space, where*

$$c_0 = \left\{ (a_i)_{i=1}^{\infty} : \lim_{i \to \infty} a_i = 0 \right\}$$

and $\|(a_i)\|_0 = \sup_{i \in \mathbb{N}} |a_i|$.

The proof that ℓ_p and c_0 are Banach spaces can be found in [5]. We need the next two definitions for the ease of notation later.

Definition 8. *The space c_{00} is the vector space of all infinite sequences of real numbers whose support is finite.*

Definition 9. *For $x \in c_{00}$, let $x(i)$ be the i^{th} coordinate of x. The standard unit vectors $(e_i)_{i=1}^{\infty}$ of c_{00} are defined by $e_i(i) = 1$ and $e_i(j) = 0$ for all $j \neq i$.*

2.2 ISOMETRIES

Definition 10. *Given a Banach space X, an isometry of X is an onto, linear mapping $U : X \to X$ such that for all $x \in X$, $||Ux|| = ||x||$; that is, U is norm-preserving.*

Definition 10 guarantees that an isometry is a bijective mapping. To see why, suppose an isometry U maps two vectors x and y to a common vector z. Then $||x - y|| = ||U(x - y)|| = ||Ux - Uy|| = ||z - z|| = 0$, which implies $x = y$. Therefore, U is injective. As U is defined to be onto, U is bijective.

The next proposition is self-evident.

Proposition 11. *If $U : X \to X$ is an isometry, then $U^{-1} : X \to X$ is an isometry.*

Proposition 12. *Isometries map extreme points to extreme points.*

Proof. Let X be a Banach space, $x \in E(X)$ and U be an isometry. Write $Ux = \frac{1}{2}y + \frac{1}{2}z$ for $y, z \in B(X)$. We have $x = \frac{1}{2}U^{-1}y + \frac{1}{2}U^{-1}z$. Since U^{-1} is an isometry, $U^{-1}y, U^{-1}z \in B(X)$. Because x is an extreme point, $U^{-1}y = U^{-1}z = x$. This proves that $y = z$ and so, $Ux \in E(X)$. $\square$

2.3 THE $\lambda-$PROPERTY

Definition 13. *A Banach space X is said to have the $\lambda-$property if for all x in $B(X)$, there exists $0 < \lambda \leq 1$ such that $x = \lambda e + (1-\lambda)y$ for some $e \in E(X)$, $y \in B(X)$. When a vector x can be written in terms of λ, e, y, we denote $(e, y, \lambda) \sim x$.*

Note that for a non-zero $x \in B(X)$ we have

$$x = \frac{1}{2}\frac{x}{\|x\|} + \frac{1}{2}(2\|x\| - 1)\frac{x}{\|x\|}.$$

Consequently, in order to verify that X has the $\lambda-$property it suffices to show that for each $x \in S(X)$ there is $(e, y, \lambda) \in E(X) \times B(X) \times (0, 1]$ with $x \sim (e, y, \lambda)$.

If X has the $\lambda-$property, for a vector x, we may find different triples (e, y, λ) such that $(e, y, \lambda) \sim x$. This leads us to define the following function: Given $x \in B(X)$,

$$\lambda(x) = \sup\{\lambda : (e, y, \lambda) \sim x \text{ for some } e, y\}.$$

Definition 14. *If there exists $\lambda_0 > 0$ such that for all $x \in B(X), \lambda(x) \geq \lambda_0$, we say that X has the uniform $\lambda-$property.*

It is trivial to show that the unit ball of the space c_0 has no extreme points. Therefore, c_0 does not have the $\lambda-$property. We consider ℓ_1.

Proposition 15.

$$E(\ell_1) = \{\pm e_i | i \in \mathbb{N}\}.$$

Proof. Let $x \in E(\ell_1)$. If there exists $0 < |x(j)| < 1$, then there must exist $0 < |x(k)| < 1$ for some $k \neq j$ because $||x|| = 1$. Form

$$x_1 = (\varepsilon \cdot \text{sign}(x(j)) + x(j))e_j + (-\varepsilon \cdot \text{sign}(x(k)) + x(k))e_k + \sum_{i \neq j,k} x(i)e_i,$$

where $0 < \varepsilon < 1 - \min\{1 - |x(j)|, 1 - |x(k)|\}$. Form $x_2 = 2x - x_1$; equivalently, $x = \frac{1}{2}(x_1 + x_2)$. Due to the way we pick ε, $||x_1|| = ||x_2|| = 1$ and $x_1 \neq x_2$, which contradicts that $x \in E(\ell_1)$. Hence, for all $x \in E(\ell_1)$, there is no j such that $0 < |x(j)| < 1$. Because $||x|| = 1$, $x = \pm e_i$ for some $i \in \mathbb{N}$.

It suffices to show that for all $i \in \mathbb{N}$, $e_i \in E(\ell_1)$. Suppose that $e_i = \frac{1}{2}(e_{i,1} + e_{i,2})$ with $e_{i,1}, e_{i,2} \in B(\ell_1)$. Without loss of generality, assume that $e_{i,1}(i) = e_i(i) + \varepsilon = 1 + \varepsilon$ and $e_{i,2}(i) = e_i(i) - \varepsilon = 1 - \varepsilon$ for some $\varepsilon \geq 0$. Because $e_{i,1} \in B(\ell_1)$, $\varepsilon = 0$. Hence, $e_{i,1}(i) = e_{i,2}(i) = 1$, which implies that $e_{i,1} = e_{i,2} = e_i$. Therefore, e_i is an extreme point. $\square$

The next proposition results directly from Theorem 1.11 in [3].

Proposition 16. *The ℓ_1 space has the $\lambda-$property but not the uniform $\lambda-$property.*

Proof. Given $x \in S(\ell_1)$, we show that x can be written as $\lambda x_1 + (1 - \lambda)x_2$ with $\lambda > 0, x_1 \in E(\ell_1), x_2 \in B(\ell_1)$. Pick $k \in \mathbb{N}$ such that $|x(k)| > 0$. If $|x(k)| = 1$, $x \in E(\ell_1)$ and we are done. If $|x(k)| \neq 1$, we write:

$$\begin{aligned} x &= \sum_i x(i)e_i = \sum_i |x(i)| \cdot \mathrm{sign}(x(i))e_i \\ &= |x(k)| \cdot \mathrm{sign}(x(k))e_k + \sum_{i \neq k} |x(i)| \cdot \mathrm{sign}(x(i))e_i \\ &= |x(k)| \cdot \mathrm{sign}(x(k))e_k + \\ &\quad (1 - |x(k)|)\left(\sum_{i \neq k} \frac{|x(i)|}{1 - |x(k)|} \cdot \mathrm{sign}(x(i))e_i\right). \end{aligned}$$

Because

$$\begin{aligned} \left|\left|\sum_{i \neq k} \frac{|x(i)|}{1 - |x(k)|} \cdot \mathrm{sign}(x(i))e_i\right|\right| &\leq \sum_{i \neq k} \left|\left|\frac{|x(i)|}{1 - |x(k)|} \cdot \mathrm{sign}(x(i))e_i\right|\right| \\ &= \sum_{i \neq k} \frac{|x(i)|}{1 - |x(k)|} = 1, \end{aligned}$$

we have written x as $\lambda x_1 + (1 - \lambda)x_2$ with $\lambda > 0, x_1 \in E(\ell_1)$, $x_2 \in B(\ell_1)$.

Next, we prove that ℓ_1 does not have the uniform $\lambda-$property. We use proof by contradiction. Suppose that ℓ_1 has the uniform $\lambda-$property; that is, there exists a $\lambda_0 > 0$ such that for all $x \in B(\ell_1)$, $\lambda(x) \geq \lambda_0$. Because $B(\ell_1) \neq E(\ell_1)$, $0 < \lambda_0 < 1$. Let $k = \lfloor 3/\lambda_0 \rfloor$ and form $x \in B(\ell_1)$ such that for all $1 \leq i \leq k$, $x(i) = \lambda_0/3$, and for $i > k$, $x(i) = 0$. Denote $L = \{\lambda : (e, y, \lambda) \sim x\}$.

1. Case 1: there exists $\lambda' \in L$ such that $\lambda' \geq \lambda_0$. Then we can write $x = \lambda' x_1 + (1 - \lambda')x_2$ with $x_1 \in E(\ell_1), x_2 \in B(\ell_1), \lambda' \geq \lambda_0$. So, $x_2 = \frac{x - \lambda' x_1}{1 - \lambda'}$. Due to the way we build x and $\lambda' \geq \lambda_0$, it is easy to see that x_2 has exactly one coordinate different from and greater than the corresponding coordinate in x, and so $||x - \lambda' x_1|| > 1$. Therefore, $||x_2|| > 1$, which contradicts that $x_2 \in B(\ell_1)$.

2. Case 2: there is no $\lambda' \in L$ such that $\lambda' \geq \lambda_0$. Because $\sup L = \lambda_0$, there exists a sequence $(\lambda_i)_{i=1}^{\infty} \subseteq L$ such that $\lambda_i < \lambda_0$ and $\lim_{i \to \infty} \lambda_i = \lambda_0$. Pick λ_n such that $\lambda_0 - \lambda_n < \lambda_0/10$ or, equivalently, $\lambda_n > \frac{9}{10}\lambda_0$. As above, it is easy to see that $||x - \lambda_n x_1|| > 1$ because x_2 has exactly one coordinate different from and greater than the corresponding coordinate in x. Therefore, $||x_2|| > 1$, which contradicts that $x_2 \in B(\ell_1)$.

This completes our proof that ℓ_1 has the $\lambda-$property but does not have the uniform $\lambda-$property. □

If $1 < p < \infty$ and $x, y \in S(\ell_p)$ we have that $\|x + y\| < 2$ when $x \neq y$. Therefore $E(\ell_p) = S(\ell_p)$ and the following holds.

Proposition 17. *For $1 < p < \infty$, ℓ_p has the uniform $\lambda-$property with $\lambda_0 = \frac{1}{2}$.*

Part II

THE SCHREIER SPACE

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THE SCHREIER SPACE $X_{\mathcal{S}_1}$ AND ITS $\lambda-$PROPERTY

Definition 18. *A set $F \subseteq \mathbb{N}$ is a Schreier set if $|F| \leq \min F$.*

For example, $\{2\}, \{2,5\}, \{3,4,7\}$ are Schreier sets, but $\{2,3,4\}$ is not. Denote $\mathcal{S}_1 = \{F : |F| \leq \min F\}$.

Definition 19. *A set F is called non-maximal if $|F| < \min F$.*

The Banach space $X_{\mathcal{S}_1}$ is defined as the completion of c_{00} with respect to the following norm:

$$\|x\|_{\mathcal{S}_1} = \sup_{F \in \mathcal{S}_1} \sum_{i \in F} |x(i)|.$$

Though the Schreier space has been studied extensively in [10], [14] and [7], there is still much unknown. The next several chapters prove its $\lambda-$property [14], find its isometries (new result), and partially characterize the extreme points of its unit ball.

In this chapter, we present the proof of the $\lambda-$property for the Schreier space. This result is first proved by Shura and Trautman in [14]. Though we use the same argument, we present the proof here for two reasons. First, the proof in [14] did not explain several points fully, and second, understanding this proof makes it easier to understand our proof of the uniform $\lambda-$property for the higher order spaces.

We call $F \in \mathcal{S}_1$ a 1-set for $x \in S(X_{\mathcal{S}_1})$ if $\sum_{i \in F} |x(i)| = 1$ and $|x(i)| > 0$ for all $i \in F$. Let $\mathcal{S}_1^x$ denote the set of all $1-$sets of x. Let $\mathcal{A}_1^x = \{F \in \mathcal{S}_1 : \sum_{i \in F} |x(i)| = 1\}$. Clearly, $\mathcal{S}_1^x \subseteq \mathcal{A}_1^x$ and $x \in S(X_{\mathcal{S}_1})$ has only maximal 1-sets if and only if $\mathcal{S}_1^x = \mathcal{A}_1^x$. We will need the following classical result of Carathéodory.

Proposition 20. *Let X be an $n-$dimensional normed space. Every $x \in B(X)$ is the convex combination of at most $n+1$ extreme points of $B(X)$.*

The following lemma is proved as Lemma 2.4 and Lemma 2.5 in [6].

Lemma 21. *Given $x \in S(X_{\mathcal{S}_1})$,*

1. *The set $\mathcal{S}_1^x$ is finite.*

2. *There exists $\varepsilon_x > 0$ such that for all $F \in \mathcal{S}_1 \backslash \mathcal{A}_1^x$,*
$$\sum_{i\in F} |x(i)| \;<\; 1-\varepsilon_x.$$
We call ε_x the $\varepsilon-$gap of x.

Note that the vector
$$x = \left(1, \frac{1}{2}, \frac{1}{2}, \underbrace{\frac{1}{4}, \ldots, \frac{1}{4}}_{4}, \underbrace{\frac{1}{8}, \ldots, \frac{1}{8}}_{8}, \ldots\right)$$
seems to contradict item 1 above. However, $x \notin X_{\mathcal{S}_1}$ because the sequence $(||x - \sum_{i\leq N} x(i)||)_{N=1}^{\infty}$ is not a Cauchy sequence.

The following lemma plays a key role in our argument. We denote $F_{<N} = F \cap [1, N]$.

Lemma 22. *Let $x \in S(X_{\mathcal{S}_1})$.*

1. *There exist $x_1, x_2 \in S(X_{\mathcal{S}_1})$ with $x_1 \in c_{00}$ and $x = \frac{1}{2}(x_1 + x_2)$.*
2. *If $x \in c_{00}$, there exist $x_1, x_2 \in S(X_{\mathcal{S}_1}) \cap c_{00}$ so that both x_1 and x_2 have non-maximal 1-sets and $x = \frac{1}{2}(x_1 + x_2)$.*
3. *If x has a 1-set F, and $x = \sum_k \lambda_k x_k$, where for all k, $x_k \in S(X_{\mathcal{S}_1})$, $\lambda_k > 0$, and $\sum_k \lambda_k = 1$, then F is a 1-set for each x_k.*

Proof. We prove item 1. If $x \in c_{00}$, then we set $x_1 = x_2 = x$ and we are done. If $x \notin c_{00}$; that is, x is infinitely supported, we pick $N \in \mathbb{N}$ such that $N > \max\{\max F : F \in \mathcal{S}_1^x\}$ and $|| \sum_{i>N} x(i)e_i|| < \varepsilon_x/2$. We form two vectors x_1 and x_2 as follows: $x_1 = \sum_{i=1}^{N} x(i)e_i$ and $x_2 = 2x - x_1$. Clearly, $x = \frac{1}{2}(x_1 + x_2)$ and $||x_1|| \leq ||x|| = 1$. It suffices to show that $||x_2|| \leq 1$. Let $F \in \mathcal{S}_1$ such that $x_2(i) \neq 0$ for all $i \in F$. If $\max F \leq N$, $\sum_{i\in F} |x_2(i)| \leq ||x|| = 1$. If $\min F > N$, $\sum_{i\in F} |x_2(i)| \leq 2 \cdot || \sum_{i>N} x(i)e_i||$ $< 2 \cdot \varepsilon_x/2 = \varepsilon_x$. The only case left is when $\max F > N$ and $\min F \leq N$. For this case, we write
$$\begin{aligned}\sum_{i\in F} |x_2(i)| \;&=\; \sum_{i\in F, i\leq N} |x_2(i)| + \sum_{i\in F, i>N} |x_2(i)| \\ &<\; (1-\varepsilon_x) + 2\cdot \varepsilon_x/2 \;=\; 1.\end{aligned}$$
So, $||x_2|| \leq 1$. Because $1 = ||x|| = ||\frac{1}{2}(x_1 + x_2)|| \leq \frac{1}{2}||x_1|| + \frac{1}{2}||x_2|| \leq \frac{1}{2} + \frac{1}{2} = 1$, $||x_1|| = ||x_2|| = 1$.

Next, we prove item 2. If x has a non-maximal 1-set, we are done by setting $x_1 = x_2 = x$. Suppose that all 1-sets of x are maximal. Let $N = \max \operatorname{supp} x + 1$. Pick $M > N$ such that $\frac{N-2}{M} < \varepsilon_x$. We form x_1 and x_2 as follows:
$$\begin{cases} x_1(i) \;=\; x(i) \text{ for all } i \leq M \\ x_1(i) \;=\; \frac{1}{M} \text{ for } M+1 \leq i \leq 2M \\ x_1(i) \;=\; 0 \text{ for } i \geq 2M+1, \end{cases}$$

and

$$\begin{cases} x_2(i) &= x(i) \text{ for all } i \leq M \\ x_2(i) &= -\frac{1}{M} \text{ for } M+1 \leq i \leq 2M \\ x_2(i) &= 0 \text{ for } i \geq 2M+1. \end{cases}$$

We see that x_1, x_2 have the non-maximal 1-set $\{M+1, M+2, \ldots, 2M\}$, and $x = \frac{1}{2}(x_1 + x_2)$. It suffices to prove that $||x_1|| \leq 1$. Let $F \in \mathcal{S}_1$. If $\max F \leq M$, $\sum_{i\in F} |x_1(i)| \leq ||x|| = 1$. If $\min F \geq N$, $\sum_{i\in F} |x(i)| \leq M \cdot \frac{1}{M} = 1$. The only case left is $\max F > M$ and $\min F < N$. We write

$$\begin{aligned} \sum_{i\in F} |x_1(i)| &= \sum_{i\in F, i<N} |x_1(i)| + \sum_{i\in F, i\geq N} |x_1(i)| \\ &< (1-\varepsilon_x) + \frac{N-2}{M} < 1 - \varepsilon_x + \varepsilon_x = 1. \end{aligned}$$

The reason $\sum_{i\in F, i<N} |x_1(i)| < 1 - \varepsilon_x$ is that x does not have a non-maximal 1-set and so, $F_{<N} \notin \mathcal{A}_1^x$. We have show that $||x_1|| \leq 1$; similarly, $||x_2|| \leq 1$, and because $x = \frac{1}{2}(x_1 + x_2)$, $||x_1|| = ||x_2|| = 1$. This completes our proof of item 2.

Finally, we prove item 3. Let $F \in \mathcal{S}_1$ be a 1-set of x. We have

$$\begin{aligned} 1 = \sum_{i\in F} |x(i)| = \sum_{i\in F} |\sum_k \lambda_k x_k(i)| &\leq \sum_k \lambda_k \sum_{i\in F} |x_k(i)| \\ &\leq \sum_k \lambda_k \cdot 1 = 1. \end{aligned}$$

Therefore, $\sum_{i\in F} |x_k(i)| = 1$ for all k or F is a 1-set for all x_k. This completes our proof of item 3. □

Let $X_{\mathcal{S}_1,n}$ be the $n-$dimensional Schreier space; that is, for all $x \in X_{\mathcal{S}_1,n}$, $x(i) = 0$ for all $i \geq n+1$.

Lemma 23. *Let $x \in E(X_{\mathcal{S}_1,n})$. If x has a non-maximal 1-set, then $x \in E(X_{\mathcal{S}_1})$.*

Proof. Let $x \in E(X_{\mathcal{S}_1,n})$ and x has a non-maximal 1-set F. Suppose that $x = \frac{1}{2}(x_1 + x_2)$, where $x_1, x_2 \in B(X_{\mathcal{S}_1})$. By Lemma 22 item 3, x_1 and x_2 has the same non-maximal 1-set as x. Therefore, $x_1, x_2 \in X_{\mathcal{S}_1,n}$. Because $x \in E(X_{\mathcal{S}_1,n})$, $x_1 = x_2 = x$ and so, $x \in E(X_{\mathcal{S}_1})$. □

Theorem 24. *The Schreier space $X_{\mathcal{S}_1}$ has the $\lambda-$property.*

Proof. We tie all the results we have shown to prove the $\lambda-$property for $X_{\mathcal{S}_1}$. Let $x \in S(X_{\mathcal{S}_1})$. By Lemma 22 item 1, we can write $x = \frac{1}{2}(x_1 + x_2)$, where $x_1, x_2 \in S(X_{\mathcal{S}_1}), x_1 \in c_{00}$. By Lemma 22 item 2, we write $x_1 = \frac{1}{2}(x_{1,1} + x_{1,2})$, where $x_{1,1}, x_{1,2} \in S(X_{\mathcal{S}_1})$, and $x_{1,1}, x_{1,2}$ have a non-maximal 1-set. Because $x_{1,1} \in c_{00}$, we know that $x_{1,1} \in B(X_{\mathcal{S}_1,n})$ for some n. According to Proposition 20,

$$x_{1,1} = \sum_{i=1}^{m} \lambda_i y_i,$$

where for all $1 \leq i \leq m$, $y_i \in E(X_{\mathcal{S}_1,n})$, $\lambda_i > 0$, and $\sum_{i=1}^m \lambda_i = 1$.

By Lemma 22 item 3, for all $1 \leq i \leq m$, y_i has the same non-maximal 1-set as $x_{1,1}$. Lemma 23 guarantees that $y_i \in E(X_{\mathcal{S}_1})$ for all $1 \leq i \leq m$. We have written

$$\begin{aligned} x &= \frac{1}{4}x_{1,1} + \frac{1}{4}x_{1,2} + \frac{1}{2}x_2 = \frac{1}{4}\sum_{i=1}^m \lambda_i y_i + \frac{1}{4}x_{1,2} + \frac{1}{2}x_2 \\ &= \frac{\lambda_1}{4}y_1 + \frac{1}{4}\sum_{i=2}^m \lambda_i y_i + \frac{1}{4}x_{1,2} + \frac{1}{2}x_2 \\ &= \frac{\lambda_1}{4}y_1 + \frac{4-\lambda_1}{4}\left(\sum_{i=2}^m \frac{\lambda_i}{4-\lambda_1}y_i + \frac{1}{4-\lambda_1}x_{1,2} + \frac{2}{4-\lambda_1}x_2\right). \end{aligned}$$

Because

$$\begin{aligned} &\left|\left|\sum_{i=2}^m \frac{\lambda_i}{4-\lambda_1}y_i + \frac{1}{4-\lambda_1}x_{1,2} + \frac{2}{4-\lambda_1}x_2\right|\right| \\ &\leq \sum_{i=2}^m \frac{\lambda_i}{4-\lambda_1}||y_i|| + \frac{1}{4-\lambda_1}||x_{1,2}|| + \frac{2}{4-\lambda_1}||x_2|| \\ &= \sum_{i=2}^m \frac{\lambda_i}{4-\lambda_1} + \frac{1}{4-\lambda_1} + \frac{2}{4-\lambda_1} = 1, \end{aligned}$$

we have $x = \frac{\lambda_1}{4}y_1 + \frac{4-\lambda_1}{4}z$, where $z \in B(X_{\mathcal{S}_1})$. Because $y_1 \in E(X_{\mathcal{S}_1})$, we have shown that $X_{\mathcal{S}_1}$ has the $\lambda-$property. □

We do not know if the Schreier space has the uniform $\lambda-$property or not, and this is still an open problem. The traditional approach to prove (or disprove) the uniform $\lambda-$property involves the characterization of all extreme points of the unit ball of the space. However, as we show later, it is quite difficult to characterize extreme points of $B(X_{\mathcal{S}_1})$ fully.

4

ISOMETRIES OF THE SCHREIER SPACE

This chapter characterizes all isometries of the Schreier space. Though we generalize our proof to find all isometries of $X_{\mathcal{S}_n}$ later, the proof for $X_{\mathcal{S}_1}$ gives a better sense of the argument we use. The following is the main theorem of this chapter.

Theorem 25. *A mapping $U : X_{\mathcal{S}_1} \to X_{\mathcal{S}_1}$ is an isometry if and only if for all $i \in \mathbb{N}$, $Ue_i = \pm e_i$.*

For the rest of this chapter, we consider an isometry $U : X_{\mathcal{S}_1} \to X_{\mathcal{S}_1}$.

Lemma 26. *Either $Ue_1 = e_1$ or $Ue_1 = -e_1$.*

Proof. Let $Ue_1 = x_1$. If $\max\operatorname{supp} x_1 = 1$, we are done. Suppose that there exists $k \geq 2$ such that $x_1(k) \neq 0$. Because $x_1 \in B(X_{\mathcal{S}_1})$, there exists $n \in \mathbb{N}$ with $2|x_1(n)| < |x_1(k)|$. Consider $\varepsilon_n e_n$, where $\varepsilon_n = \begin{cases} 1 - x_1(n) \text{ if } x_1(n) \geq 0 \\ -1 - x_1(n) \text{ if } x_1(n) < 0 \end{cases}$. Let $Uz = \varepsilon_n e_n$. We have:

$$\begin{aligned} ||e_1 + z|| &= ||U(e_1+z)|| = ||Ue_1 + Uz|| = ||x_1 + \varepsilon_n e_n|| \\ &\geq |x_1(k)| + |x_1(n) + \varepsilon_n| = |x_1(k)| + 1 > 1. \end{aligned}$$

Let $F \in \mathcal{S}_1$ such that $\sum_{i\in F} |(e_1 + z)(i)| > 1$. If $1 \notin F$, $\sum_{i\in F} |(e_1 + z)(i)| \leq ||z|| \leq 1$, which is a contradiction. So, $1 \in F$ and so, $F = \{1\}$, which implies that $|1 + z(1)| > 1$. We also have:

$$\begin{aligned} ||e_1 - z|| &= ||U(e_1 - z)|| = ||Ue_1 - Uz|| = ||x_1 - \varepsilon_n e_n|| \\ &\geq |x_1(k)| + |x_1(n) - \varepsilon_n| \geq |x_1(k)| + |\varepsilon_n| - |x_1(n)| \\ &\geq |x_1(k)| + (1 - |x_1(n)|) - |x_1(n)| \\ &= 1 + (|x_1(k)| - 2|x_1(n)|) > 1. \end{aligned}$$

Similarly, we can argue that $|1 - z(1)| > 1$. Because $|z(1)| \leq 1$, we cannot have $|1 - z(1)| > 1$ and $|1 + z(1)| > 1$ at the same time. Hence, $\max\operatorname{supp} x_1 = 1$. This completes our proof. □

Lemma 27. *For all $i \in \mathbb{N}$, $Ue_i \in c_{00}$.*

Proof. The case when $i = 1$ follows from Lemma 26. Let $i \in \mathbb{N}_{\geq 2}$. Due to Proposition 12, because $e_1 + e_i \in E(X_{\mathcal{S}_1})$, $U(e_1 + e_i) = \pm e_1 + Ue_i \in E(X_{\mathcal{S}_1})$. By Lemma 30 item 1 (next chapter), $E(X_{\mathcal{S}_1}) \subseteq c_{00}$ and so, $Ue_i \in c_{00}$. □

Denote $K = \{\pm e_i | i \in \mathbb{N}\}$. Let $Ue_2 = x_2 \in c_{00}$ and $\max\operatorname{supp} x_2 = k$.

Lemma 28. *For all $i \geq k+1, Ue_i = \pm e_i$.*

Proof. Because $e_1 + x_2 \in E(X_{\mathcal{S}_1})$, x_2 has a non-maximal 1-set due to Lemma 30 item 1. Let $Uy_{k+1} = e_{k+1}$, then

$$||e_2 + y_{k+1}|| \ = \ ||x_2 + e_{k+1}|| \ = \ 2.$$

This implies that there exists some $m \geq 2$ such that $y_{k+1}(m) = \pm 1$. Because $||y_{k+1}|| = 1$, $y_{k+1} \in K$. This result does not apply only to e_{k+1}. Hence, we know that for all $i \geq (k+1)$, there exists a j such that either $Ue_j = e_i$ or $U(-e_j) = e_i$. As a result, for all $N \in \mathbb{N}$, there exist i and j such that $i, j > N$ and either $Ue_i = e_j$ or $Ue_i = -e_j$.

Pick e_ℓ with $\ell \geq k+1$ and let $Ue_n = e_\ell$ or $Ue_n = -e_\ell$ for some n. Suppose that $\ell > n$. By the above observation, we can find $\ell < \ell_1 < \ell_2 < \ldots < \ell_n$ with $\ell < k_1 < k_2 < \ldots < k_n$ such that $Ue_{k_i} = e_{\ell_i}$ or $Ue_{k_i} = -e_{\ell_i}$. By definition, we have

$$||e_n + \sum_1^n e_{k_i}|| \ = \ ||e_\ell + \sum_1^n Ue_{k_i}|| \ = \ ||e_\ell + \sum_1^n \varepsilon_i e_{\ell_i}||,$$

where $\varepsilon_i \in \{\pm 1\}$. However, the leftmost is n while the rightmost is $n+1$, which is a contradiction. If $\ell < n$, using a similar argument, we arrive at the same contradiction. This shows us that for all $i \geq k+1, Ue_i = e_i$ or $Ue_i = -e_i$. $\square$

Lemma 29. *Either $Ue_2 = e_2$ or $Ue_2 = -e_2$.*

Proof. Recall that $Ue_2 = x_2$ and $\max\operatorname{supp} x_2 = k > 2$. By isometry, we have:

$$||e_2 + \sum_{i=k+1}^{2k-1} e_i|| \ = \ ||U(e_2 + \sum_{i=k+1}^{2k-1} e_i)|| \ = \ ||x_2 + \sum_{i=k+1}^{2k-1} Ue_i||.$$

However, the leftmost is $k-1$, while due to Lemma 28, the rightmost is at least $|x_2(k)| + (k-1) > (k-1)$, which is a contradiction. Therefore, $\max\operatorname{supp} x_2 = 2$. $\square$

Proof of Theorem 25. The forward direction of Theorem 25 follows immediately from Lemma 26, Lemma 28, and Lemma 29. The backward direction is clearly true. $\square$

5

EXTREME POINTS OF $B(X_{\mathcal{S}_1})$

This chapter presents what we know about the extreme points of the unit ball of the Schreier space. In [14], the authors claim (without a proof) that if $x \in E(X_{\mathcal{S}_1})$, then $\operatorname{supp} x$ is finite and has an even cardinality. We prove a stronger result and present many new extreme points that have not been seen before.

Lemma 30. *Let $x \in E(X_{\mathcal{S}_1})$. Then*

1. *the vector x has a non-maximal 1-set,*
2. *for all $i \in \mathbb{N}$, there exists $F \in \mathcal{A}_1^x$ such that $i \in F$,*
3. *the first coordinate $x(1) = \pm 1$.*

Proof. We first prove item 1. Suppose that x does not have a non-maximal 1-set, meaning that $\mathcal{S}_1^x = \mathcal{A}_1^x$. By Lemma 21, $\mathcal{S}_1^x$ is finite. Pick $N > \max\{\max F : F \in \mathcal{S}_1\}$. Form x_1 such that $x_1(i) = x(i)$ for all $i \neq N$, and $x_1(N) = x(i) + \varepsilon_x$. Form x_2 such that $x_2(i) = x(i)$ for all $i \neq N$, and $x_2(N) = x(i) - \varepsilon_x$. Clearly, $x = \frac{1}{2}(x_1 + x_2)$. Let $F \in \mathcal{S}_1$. If $N \notin F$, $\sum_{i \in F} |x_1(i)| \leq ||x|| = 1$. If $N \in F$ (hence, $F \notin \mathcal{A}_1^x$), we have

$$\begin{aligned} \sum_{i \in F} |x_1(i)| &= \sum_{i \in F, i \neq N} |x_1(i)| + |x_1(N)| \leq \Big(\sum_{i \in F, i \neq N} |x(i)| + |x(N)|\Big) + \varepsilon_x \\ &< (1 - \varepsilon_x) + \varepsilon_x = 1. \end{aligned}$$

Therefore, $x_1 \in B(X_{\mathcal{S}_1})$ and similarly, we can show that $x_2 \in B(X_{\mathcal{S}_1})$. Because $x_1 \neq x_2$, $x \notin E(X_{\mathcal{S}_1})$, a contradiction. So, x must have a non-maximal 1-set.

Next, we prove item 2. Suppose that there exists some $N \in \mathbb{N}$ such that for all $F \in \mathcal{A}_1^x$, $N \notin F$. Hence, for all $F \in \mathcal{S}_1$ and $N \in F$, $\sum_{i \in F} |x(i)| < 1 - \varepsilon_x$. Form x_1 such that $x_1(i) = x(i)$ for all $i \neq N$, and $x_1(N) = x(i) + \varepsilon_x$. Form x_2 such that $x_2(i) = x(i)$ for all $i \neq N$, and $x_2(N) = x(i) - \varepsilon_x$. Clearly, $x = \frac{1}{2}(x_1 + x_2)$. Let $F \in \mathcal{S}_1$. If $N \notin F$, $\sum_{i \in F} |x_1(i)| \leq ||x|| = 1$. If $N \in F$, we have

$$\begin{aligned} \sum_{i \in F} |x_1(i)| &= \sum_{i \in F, i \neq N} |x_1(i)| + |x_1(N)| \leq \sum_{i \in F, i \neq N} |x(i)| + |x(N)| + \varepsilon_x \\ &< 1 - \varepsilon_x + \varepsilon_x = 1. \end{aligned}$$

Therefore, $x_1 \in B(X_{\mathcal{S}_1})$ and similarly, we can show that $x_2 \in B(X_{\mathcal{S}_1})$. Because $x_1 \neq x_2$, $x \notin E(X_{\mathcal{S}_1})$, a contradiction. So, for all $i \in \mathbb{N}$, there exists $F \in \mathcal{A}_1^x$ such that $i \in F$.

The proof of item 3 is trivial, so we omit it. $\square$

Definition 31. *Let $x \in E(X_{\mathcal{S}_1})$. Based on x, we can build y such that $y(i) \neq 0$ for all $1 \leq i \leq \max \operatorname{supp} y$ by pushing all coordinates of x to the left until there is no zero lying between non-zero coordinates. We call y the compact form of x, denoted compact-x.*

Lemma 32. *Let $x \in E(X_{\mathcal{S}_1})$. Then its compact form $y \in E(X_{\mathcal{S}_1})$.*

Proof. Let F be the non-maximal 1-set of x. Denote $\max(\operatorname{supp} x \backslash F) = m$. Let $1 < \ell < m$ be chosen. We will show that $x(\ell) \neq 0$. Assume that $x(\ell) = 0$. By Lemma 30 item 2, ℓ must be in some 1-set F'.

We know that $F \cup \{m\} \subseteq F'$ because if not, for $k \in F \cup \{m\}$ and $k \notin F'$, we can form $F'' = F' \cup \{k\} \backslash \{\ell\}$. Clearly, $F'' \in \mathcal{S}_1$ and $\sum_{i \in F''} |x(i)| > 1$, a contradiction. However, if $F \cup \{m\} \subseteq F'$,

$$\sum_{i \in F'} |x(i)| \ \geq \sum_{i \in F \cup \{m\}} |x(i)| \ = \ |x(m)| + \sum_{i \in F} |x(i)| \ = \ |x(m)| + 1 \ > \ 1,$$

which is a contradiction. Therefore, $x(\ell) \neq 0$.

Denote $F = \{k_1, k_2, \ldots, k_{|F|}\}$, where $k_1 < k_2 < \ldots < k_{|F|}$. We build y as follows:

$$\begin{cases} y(i) \ = \ x(i) \text{ for all } i \leq m, \\ y(m+1) \ = \ x(k_1), y(m+2) \ = \ x(k_2), \ldots, y(m+|F|) \ = \ x(k_{|F|}), \\ y(m+|F|+i) \ = \ 0 \text{ for all } i \geq 1. \end{cases}$$

It is easy to see that $y \in E(X_{\mathcal{S}_1})$. Indeed, a proof by contradiction assumes that $y \notin E(X_{\mathcal{S}_1})$ and leads to $x \notin E(X_{\mathcal{S}_1})$, a contradiction. This completes our proof. $\square$

Lemma 33. *Let $x \in E(X_{\mathcal{S}_1})$ and F be the non-maximal 1-set of x. Then $|\operatorname{supp} x| = 2|F|$.*

Proof. Let $y =$ compact-x with G being the non-maximal 1-set of y. Clearly, $\operatorname{supp} y = \operatorname{supp} x$. Let $m = \max(\operatorname{supp} y \backslash G)$ and let $G = \{m+1, m+2, \ldots, m+k\}$ for some $k \in \mathbb{N}$. If $k < m$, $\{m\} \cup G \in \mathcal{S}_1$, and $\sum_{i \in \{m\} \cup G} |y(i)| = \sum_{i \in G} |y(i)| + |y(m)| = 1 + |y(m)| > 1$. If $k > m$, then G is not a non-maximal 1-set. Therefore, $k = m$, and this completes our proof. $\square$

Corollary 34. *If $x \in E(X_{\mathcal{S}_1})$, then x has an even number of nonzero coordinates. In other words, $|\operatorname{supp} x|$ is even.*

Corollary 34 is claimed by Shura and Trautman in [14], but the authors did not give a proof.

Next, we move on to give examples of extreme points in $B(X_{\mathcal{S}_1})$. The following proposition says that if a vector x is not an extreme point we can find two distinct norm-one vectors that are as close as we wish so that x is the midpoint of the two vectors.

Theorem 35. *All vectors x created by the following process are extreme points in the unit ball of the Schreier space.*

1. *Let n be the cardinality of the non-maximal 1-set of x. Set $x(n+1) = x(n+2) = \ldots = x(2n) = 1/n$. Choose $k \in [2,n]$ and set $x(k) = (n+1-k)/n$,*
2. *For all $k+1 \leq i \leq n, x(i) = 1/n$,*
3. *For all $2 \leq i \leq k-1$, $x(i)$ forms a sum of 1 with the $i-1$ maximum values in $\{x(j) : j \geq i+1\}$.*

Example 1. Several extreme points constructed by the method are:

1. $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, \ldots\right)$,
2. $\left(1, \frac{2}{5}, \frac{3}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0, \ldots\right)$.

Are there any other forms of extreme points in the unit ball of the Schreier space? The above construction may suggest that the coordinates in the non-maximal 1-set are equal. However, we found a counterexample. The following is an extreme point:

$$x = \left(1, \frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{3}, \frac{1}{3}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, 0, \ldots\right).$$

As we have seen, $E(X_{\mathcal{S}_1})$ is difficult to characterize. In particular, the coordinates of a vector x are not necessarily increasing or decreasing (Example 1 item 1), and the coordinates corresponding to the non-maximal 1-set are not necessarily equal.

Part III

THE HIGHER ORDER SCHREIER SPACES

6

HIGHER ORDER SCHREIER SPACES AND THE $\lambda-$PROPERTY

We now present a generalization of the Schreier space. Given two sets E, F, we write $E < F$ if $\max E < \min F$, and we write $n < E$ if $n < \min E$. We define the Schreier families as follows. Letting $\mathcal{S}_0 = \{F : |F| \leq 1\}$ and supposing that $\mathcal{S}_n$ $(n \in \mathbb{N} \cup \{0\})$ has been defined, we define

$$\mathcal{S}_{n+1} \ = \ \{\bigcup_{i=1}^{n} E_i : n \leq E_1 < E_2 < ... < E_n \text{ are in } \mathcal{S}_n\}.$$

Given $\mathcal{S}_n$ and $F \in \mathcal{S}_n$, F is called non-maximal if given $m > \max F$, $F \cup \{m\} \in \mathcal{S}_n$. As in the case of the Schreier space, non-maximal sets are crucial in our later arguments. A set is maximal if it is not non-maximal. Let $\mathcal{S}_n^{MAX}$ denote the set of all maximal sets in $\mathcal{S}_n$. For each $\mathcal{S}_n$, we define the Banach space $X^p_{\mathcal{S}_n}$ as the completion of c_{00} with respect to the following norm: for $p \in [1, \infty)$,

$$\|x\|_{\mathcal{S}_n,p} \ = \ \sup_{F\in\mathcal{S}_n}(\sum_{i\in F}|x(i)|^p)^{\frac{1}{p}}.$$

Note that the Schreier space is $X^1_{\mathcal{S}_1}$. Because we generalize the Schreier space in two dimensions, which are higher order Schreier sets and $p-$convexification, our notation gets more complicated. We call $F \in \mathcal{S}_n$ a 1-set for $x \in S(X^p_{\mathcal{S}_n})$ if $(\sum_{i\in F}|x(i)|^p)^{\frac{1}{p}} = 1$ and $x(i) \neq 0$ for any $i \in F$. Let $\mathcal{S}^x_{n,p}$ be the set of all 1-sets of x. Let $\mathcal{A}^x_{n,p} = \{F \in \mathcal{F} : \sum_{i\in F}|x(i)|^p = 1\}$. Note that x has only maximal 1-sets if and only if $\mathcal{A}^x_{n,p} = \mathcal{S}^x_{n,p}$.

Lemma 36. *Let $n \in \mathbb{N}$, $p \in [1, \infty)$ and $x \in S(X^p_{\mathcal{S}_n})$. The following hold:*

1. *The set $S^x_{n,p}$ is finite.*
2. *There is an $\varepsilon_x > 0$ (which we call the ε-gap for x) so that each $F \in \mathcal{S}_n \backslash \mathcal{A}^x_{n,p}$, $\sum_{i\in F}|x(i)|^p < 1 - \varepsilon_x$.*
3. *$E(X_{\mathcal{S}_n}) \subset c_{00}$*

Proof. For a vector $x = \sum_i x(i)e_i$ define $x^p = \sum_i |x(i)|^p e_i$. Observe that if $||\sum_i x(i)e_i||_{X^p_{\mathcal{S}_n}} = 1$ then $||\sum_i |x(i)|^p e_i||_{X_{\mathcal{S}_n}} = 1$. Using [6], Lemma 2.5, we can find $\varepsilon_{x^p} > 0$ so that

$$\sum_{i\in F} |x(i)|^p \ < \ 1 - \varepsilon_{x^p}$$

for all $F \in \mathcal{S}_n \backslash \mathcal{A}^{x^p}_{n,1}$. Note that $\mathcal{A}^{x^p}_{n,1} = \mathcal{A}^x_{n,p}$ and $\mathcal{S}^{x^p}_{n,1} = \mathcal{S}^x_{n,p}$. This completes our proof of item 1 and item 2.

Suppose that $x \in S(X^p_{\mathcal{S}_n}) \setminus c_{00}$. Let k with $x(k) \neq 0$ be larger than the maximum of every $F \in \mathcal{S}^x_{n,p}$. Note it is not possible for $F \cup \{k\} \in \mathcal{S}_n$ for any $F \in \mathcal{S}^x_{n,p}$. That is, $\mathcal{S}^x_{n,p}$ consists of only maximal sets. Therefore, if we consider $F \in \mathcal{S}_n$ that contains k then $F \notin \mathcal{S}^x_{n,p}$ and so

$$\sum_{i\in F} |x(i)|^p \ < \ 1 - \varepsilon_x.$$

We can therefore perturb $x(k)$ by a value less than ε_x to produce $y, z \in S(X^p_{\mathcal{S}_n})$ with $x = 1/2(y + z)$. This is the desired result. □

The following result follows the significantly stronger statement [2], Proposition 12.9.

Proposition 37. *Fix $n \in \mathbb{N}$ and $p \in [1, \infty)$. For each $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists $F \in \mathcal{S}^{MAX}_n$ with $N \leq \min F$ and a sequence non-negative of scalars $(a_i)_{i\in F}$ with $\sum_{i\in F} a^p_i = 1$ so that for each $G \in \mathcal{S}_{n-1}$, $\sum_{i\in G} a^p_i < \varepsilon$.*

Lemma 38. *Fix $n \in \mathbb{N}$ and $p \in [1, \infty)$. Consider $\mathcal{S}_n$ and $x \in S(X^p_{\mathcal{S}_n})$.*

1. *There exist $x_1, x_2 \in S(X^p_{\mathcal{S}_n})$ with $x_1 \in c_{00}$ and $x = \frac{1}{2}(x_1 + x_2)$.*
2. *Let $x \in c_{00}$, there exist $x_1, x_2 \in S(X^p_{\mathcal{S}_n}) \cap c_{00}$ so that both x_1 and x_2 have non-maximal 1-sets and $x = \frac{1}{2}(x_1 + x_2)$.*
3. *If $x \in c_{00}$, there exist $x_1, x_2 \in S(X^p_{\mathcal{S}_n}) \cap c_{00}$ so that $x = \frac{1}{2}(x_1 + x_2)$ and for each $i \leq \max \operatorname{supp} x_1$ there is an $F \in \mathcal{A}^x_{n,p}$ with $i \in F$.*

Proof. We first prove item 1. Let $x \in S(X^p_{\mathcal{S}_n})$. If $x \in c_{00}$, then we are done by letting $x_1 = x_2 = x$. If $x \notin c_{00}$, then $\mathcal{A}^x_{n,p} = \mathcal{S}^x_{n,p}$. Using Lemma 36 we can find $\varepsilon_x > 0$. Fix $N \in \mathbb{N}$ so that $||\sum_{i>N} x(i)e_i|| < \varepsilon_x/2$ and $N > \max\{\max F : F \in \mathcal{S}^x_{n,p}\}$. Let $x_1 = \sum^N_{i=1} x(i)e_i$ and $x_2 = 2x - x_1$. Clearly, $||x_1|| \leq ||x|| = 1$. It suffices to prove that $||x_2|| \leq 1$. Let $F \in \mathcal{S}_n$. If $\max F \leq N$, then $(\sum_{i\in F} |x_2(i)|^p)^{1/p} \leq ||x|| = 1$. If $\min F > N$, then $(\sum_{i\in F} |x_2(i)|^p)^{1/p} \leq 2 \cdot ||\sum_{i>N} x(i)e_i|| < 2 \cdot \varepsilon_x/2 = \varepsilon_x$. Finally, if $\min F < N$ and $\max F > N$ (and so, $F \notin \mathcal{A}^x_{n,p}$), then we have the following:

$$\begin{aligned}(\sum_{i\in F} |x_2(i)|^p)^{1/p} \ &= \ (\sum_{i\in F, i\leq N} |x(i)|^p + 2 \sum_{i\in F, i>N} |x(i)|^p)^{1/p} \\ &< \ (1 - \varepsilon_x + \varepsilon_x)^{1/p} \ = \ 1.\end{aligned}$$

Hence, $x_1, x_2 \in B(X^p_{\mathcal{S}_n})$, and since $x = \frac{1}{2}(x_1 + x_2)$ and $||x|| = 1$, we must have $x_1, x_2 \in S(X^p_{\mathcal{S}_n})$. This finishes the proof of item 1.

Let's prove item 2. We may assume that $x \in c_{00}$ has only maximal 1-sets and let $N = \max \operatorname{supp} x$. Using Proposition 37, we can find $A \in \mathcal{S}_n^{MAX}$ with $\min A > N$ and non-negative convex scalars $(a_i)_{i\in A}$ so that for all $G \in \mathcal{S}_{n-1}$,

$$\sum_{i\in G} a_i^p \ < \ \frac{\varepsilon_x}{2N}.$$

Let $i_0 = \max A$ and $F_0 = A\backslash\{i_0\}$ and $b_i^p = a_i^p/(1 - a_{i_0}^p)$ for $i \in F_0$. We can safely assume that $a_{i_0} < \frac{1}{2}$. Clearly, $(b_i)_{i\in F_0}$ are convex scalars, F_0 is non-maximal and if $G \in \mathcal{S}_{n-1}$,

$$\sum_{i\in G} b_i^p \ < \ \frac{1}{1-a_{i_0}^p}\frac{\varepsilon_x}{2N} \ < \ \frac{\varepsilon_x}{N}.$$

Let $x_1 = x + \sum_{i\in F_0} b_i e_i$ and $x_2 = x - \sum_{i\in F_0} b_i e_i$. Since x_1 and x_2 both have F_0 as a non-maxinal 1-sets we are done once we can show that $||x_1|| = ||x_2|| = 1$. Let $F \in \mathcal{S}_n$. If $\max F \le N$, $||x_1|| \le ||x|| = 1$. If $\min F > N$, $||x_1|| \le ||\sum_{i\in F_0} b_i e_i|| = 1$. If $\max F > N$ and $\min F \le N$ (and so, $F \notin \mathcal{A}^x_{n,p}$),

$$\begin{aligned}\sum_{i\in F} |x_1(i)|^p \ &< \ \sum_{i\in F, i\le N} |x_1(i)|^p + \sum_{i\in F, i>N} |x_1(i)|^p \\ &< \ 1 - \varepsilon_x + N\cdot\frac{\varepsilon_x}{N} \ = \ 1.\end{aligned}$$

The estimate $\sum_{i\in F, i>N} |x_1(i)|^p \ < \ N \cdot \frac{\varepsilon_x}{N}$ is because $\min F \le N$ and so, F can contain at most N maximal sets in $\mathcal{S}_{n-1}$. This shows that $||x_1|| \le 1$. The same proof yields $||x_2|| \le 1$, as desired. Again, since $x = \frac{1}{2}(x_1 + x_2)$ and $||x|| = 1$, we must have $x_1, x_2 \in S(X^p_{\mathcal{S}_n})$.

Finally, we prove item 3 of the lemma. Let $x \in c_{00}$ and consider the following procedure: Let $i_1 \in [1, \max \operatorname{supp} x]$ be minimum so that for all $F \in \mathcal{S}_n$, with $i_1 \in F$, $\sum_{i\in F} |x(i)| < 1$. If no such i_1 exists we are done (let $x = x_1 = x_2$). Since there are only finitely many $F \in \mathcal{S}_n$ containing i_1 with $\max F \leqslant \max \operatorname{supp} x$ we can find $F_1 \in \mathcal{S}_n$ with

$$(\sum_{i\in F_1} |x(i)|^p)^{1/p} \ = \ \sup\{(\sum_{i\in F} |x(i)|^p)^{1/p} : F \in \mathcal{S}_n,\ i_1 \in F\}.$$

Find $\delta_{i_1} > 0$ so that

$$|x(i_1) + \operatorname{sign}(x(i_1))\delta_{i_1}|^p + \sum_{i\in F_1, i\neq i_1} |x(i)|^p \ = \ 1.$$

Let $x_{1,1} = x + \operatorname{sign}(x(i_1))\delta_{i_1} e_{i_1}$ and $x_{2,1} = x - \operatorname{sign}(x(i_1))\delta_{i_1} e_{i_1}$. We shall prove that $\|x_{1,1}\| \leqslant 1$. As such we must show for each $F \in \mathcal{S}_n$, $\sum_{i\in F} |x_{1,1}(i)| \leqslant 1$. The case that $F \in \mathcal{S}_n$ and does not contain i_1 follows from the fact that $\|x\| \leqslant 1$ and so we assume $i_1 \in F$. In this case, we use the definition of F_1 to observe that

$$\begin{aligned}\sum_{i\in F}|x_{1,1}(i)|^p &= |x(i_1)+\text{sign}(x(i_1))\delta_{i_1}|^p + \sum_{i\in F, i\neq i_1}|x(i)|^p \\ &\leqslant |x(i_1)+\text{sign}(x(i_1))\delta_{i_1}|^p + \sum_{i\in F_1, i\neq i_1}|x(i)|^p = 1.\end{aligned}$$

Therefore $\|x_{1,1}\| \leqslant 1$. Since $|x_{2,1}(i_1)| \leqslant |x_{1,1}(i_1)|$ we have $\|x_{2,1}\| \leqslant 1$ and by the same reasons of the previous items, we conclude that $\|x_{1,1}\| = \|x_{2,1}\| = 1$ and also, trivially, that $x = \frac{1}{2}(x_{1,1} + x_{2,1})$. In order to produce a vector satisfying the claim we inductively apply the above procedure as follows: Find the minimum $i_2 > i_1$ in $[1, \max\text{supp}\, x]$ and so that for all $F \in \mathcal{S}_n$, with $i_2 \in F$, $\sum_{i\in F}|x(i)| < 1$. If no such i_2 exists we are done. Since there are only finitely many $F \in \mathcal{S}_n$ containing i_2 with $\max F \leqslant \max\text{supp}\, x$ we can find $F_2 \in \mathcal{S}_n$ with

$$(\sum_{i\in F_2}|x_{1,1}(i)|^p)^{1/p} = \sup\{(\sum_{i\in F}|x_{1,1}(i)|^p)^{1/p} : F \in \mathcal{S}_n,\ i_2 \in F\}.$$

Find $\delta_{i_2} > 0$ so that

$$|x_{1,1}(i_2) + \text{sign}(x_{1,1}(i_2))\delta_{i_1}|^p + \sum_{i\in F_2, i\neq i_1}|x_{1,1}(i)|^p = 1.$$

Let $x_{1,2} = x_{1,1} + \text{sign}(x(i_2))\delta_{i_2}e_{i_2}$ and $x_{2,2} = x_{1,2} - \text{sign}(x(i_2))\delta_{i_2}e_{i_2}$. Arguing as before we have that $\|x_{1,2}\| \leqslant 1$, $\|x_{2,2}\| \leqslant 1$ and $x = \frac{1}{2}(x_{1,2} + x_{2,2})$. This procedure can be iterated the finitely many times it takes to exhaust $\text{supp}\, x$ in order to produce $x_{1,n}$ and $x_{2,n}$ with $\|x_{1,n}\| = 1$, $\|x_{2,n}\| = 1$ and $x = \frac{1}{2}(x_{1,n} + x_{2,n})$ so that $x_{1,n}$ has the property that for each $i \leqslant \max\text{supp}\, x_{1,n}$ there is an $F \in \mathcal{A}_{n,p}^{x_{1,n}}$ with $i \in F$. This yields the desired decomposition. □

We make one easy remark before proceeding. The remark is a generalized version of Lemma 22 item 3.

Remark 39. Let $x \in S(X_{\mathcal{S}_n})$ and $x = \sum_{i\in F}\lambda_i x_i$ for $x_i \in S(X_{\mathcal{S}_n})$ and convex scalars $(\lambda_i)_{i\in F}$. Then $\mathcal{A}_{n,p}^x \subset \mathcal{A}_{n,p}^{x_i}$ for each $i \in F$. This follows from triangle inequality, the fact that the scalars are convex, and $\|x_i\| \leqslant 1$ for each $i \in F$.

The next proposition is a characterization of extreme points for $B(X_{\mathcal{S}_n}^p)$ and $p \in (1, \infty)$. Such a characterization seems necessary in order to show a space has the uniform λ-property.

Proposition 40. *Let $\mathcal{S}_n$, $p \in (1,\infty)$ and $x \in S(X_{\mathcal{S}_n}^p)$. Then $x \in E(X_{\mathcal{S}_n}^p)$ if and only if $x \in c_{00}$, $\mathcal{A}_{n,p}^x$ has a non-maximal set and for all $i \leqslant \max\text{supp}\, x$ there is an $F \in \mathcal{A}_{n,p}^x$ with $i \in F$. Moreover if $p = 1$ then the forward implication holds.*

Proof. We first prove the reverse implication. Suppose $x \in c_{00}$ and satisfies the assumptions. Let $x = 1/2(z+y)$ and $F \in \mathcal{A}^x_{n,p}$. Then $\sum_{i \in F} |x(i)|^p = 1$. Since every element of the sphere of $\ell_p^{|F|}$ is an extreme point, we know in order for $\sum_{i \in F} |y(i)|^p = \sum_{i \in F} |z(i)|^p = 1$ we must have $x(i) = y(i) = z(i)$ for all $i \in F$. Our assumption is that all $i \leqslant \max\operatorname{supp} x$ is contained in a set $F \in \mathcal{A}^x_{n,p}$. Therefore $x(i) = y(i) = z(i)$ for all such $i \leqslant \max\operatorname{supp} x$. Now let $i > \max\operatorname{supp} x$. Find a non-maximal $F \in \mathcal{A}^x_{n,p}$, then $F \cup \{i\} \in \mathcal{A}^x_{n,p}$ and consequently, $x(i) = y(i) = z(i)$ by the same reasoning as above. Therefore, $z = y = x$, which implies that $x \in E(X^p_{\mathcal{S}_n})$.

We now prove the forward implication as well as the 'moreover' statement. Let $x \in S(X^p_{\mathcal{S}_n})$ for $p \in [1, \infty)$. First, Lemma 36 states that $E(X^p_{\mathcal{S}_n})$ is a subset of c_{00}. We can assume that either every set in $\mathcal{A}^x_{n,p}$ is maximal or there is an $i \leqslant \max\operatorname{supp} x$ not contained in any $F \in \mathcal{A}_x$. In the former case we have $\mathcal{A}^x_{n,p} = \mathcal{S}^x_{n,p}$ and since $\mathcal{S}^x_{n,p}$ is finite there is a $k > \max\{\max F : F \in \mathcal{S}^x_{n,p}\}$. We can perturb $x(k)$ by any value $\delta > 0$ with $\delta < \varepsilon_x$ and create new vectors $y = x - \delta x(k) e_k$ and $z = x + \delta x(k) e_k$ that are in $S(X^p_{\mathcal{F}})$ and satisfy $x = 1/2(y+z)$. In the later case, we can find the coordinate $k \leqslant \max\operatorname{supp}\, x$ and similarly show that x is not an extreme point. □

Theorem 41. *Let $n \in \mathbb{N}$*

1. *For $p \in (1, \infty)$, the space $X^p_{\mathcal{S}_n}$ has the uniform $\lambda-$property.*
2. *The space $X_{\mathcal{S}_n}$ has the $\lambda-$property.*

Proof. First, we prove item 1. Let $x \in S(X^p_{\mathcal{S}_n})$ for $p \in (1, \infty)$. Using Lemma 38 item 1, we can find $x_1 \in c_{00}$ and $x_1, x_2 \in S(X^p_{\mathcal{S}_\alpha})$ and so that $x = 1/2(x_1 + x_2)$. Now apply Lemma 38 item 2, to find $x_{1,1}$ and $x_{1,2}$ in $c_{00} \cap S(X^p_{\mathcal{S}_n})$ each with a non-maximal $1-$set so that $x_1 = 1/2(x_{1,1} + x_{1,2})$. Finally, we apply Lemma 36 item 3 to find $x_{1,1,1}$ and $x_{1,1,2}$ in $c_{00} \cap S(X^p_{\mathcal{S}_n})$ with $x_{1,1} = 1/2(x_{1,1,1} + x_{1,1,2})$ so that $x_{1,1,1}$ has both a non-maximal 1-set and for each $i \leqslant \max\operatorname{supp}\, x_{1,1,1}$ there is an $F \in \mathcal{A}^{x_{1,1,1}}_{n,p}$ with $i \in F$. Proposition 40 implies that $x_{1,1,1} \in E(X^p_{\mathcal{S}_n})$. Therefore X has the uniform $\lambda-$property as

$$x = \frac{1}{8} x_{1,1,1} + \frac{1}{8} x_{1,1,2} + \frac{1}{4} x_{1,2} + \frac{1}{2} x_2.$$

We now prove item 2. The beginning of the proof is the same, however, we are not able to conclude that $x_{1,1,1}$ is an extreme point. We do know, however, that $x_{1,1,1}$ is finitely supported with a non-maximal 1-set. Therefore there is an $n \in \mathbb{N}$ so that $x_{1,1,1} \in \operatorname{span}\{e_1, \cdots, e_n\}$. By Carathéodory's Theorem, every point of the unitary ball of an $n-$dimensional normed space is the convex combination of at most

$n+1$ many extreme points of the ball. Hence, there are a $d \leqslant n+1$ and extreme points $(y_i)_{i=1}^d$ of $B(\text{span}\{e_1, \cdots, e_n\})$ so that

$$x_{1,1,1} = \sum_{i=1}^{d} \lambda_i y_i$$

with $\sum_{i=1}^d \lambda_i = 1$ and $\lambda_i > 0$. By Remark 39, $\mathcal{A}_{n,p}^{x_{1,1,1}} \subseteq \mathcal{A}_{n,p}^{y_i}$ and so, each y_i has the same non-maximal 1-set F as $x_{1,1,1}$. It follows that each y_i is an extreme point of $X_{\mathcal{S}_n}$ as well. Indeed, if $y_i = 1/2(z+w)$ for $z, w \in B(X_{\mathcal{S}_n})$, then $z(k) = w(k) = 0$ for all $k > n$. Suppose not; that is, there exists $z(k_0) \neq 0$, then $||z|| \geq \sum_{i \in F \cup \{k_0\}} |z(i)| > 1$. Since y_i is in extreme point of $B(\text{span}\{e_1, \cdots, e_n\})$ and $z, w \in B(\text{span}\{e_1, \cdots, e_n\})$, $z = w = y_i$. This implies that y_i is in $E(X_{\mathcal{S}_n})$ and so $X_{\mathcal{S}_n}$ has the $\lambda-$property. $\square$

7

ISOMETRIES OF HIGHER ORDER SCHREIER SPACES

In this section, we will use our previous results concerning extreme points of Schreier space to exhibit the general form of the elements of $\operatorname{Isom}(X_{\mathcal{S}_n})$, with $n \in \mathbb{N}$. We state the main result.

Theorem 42. *Let $n \in \mathbb{N}$ and $U \in \operatorname{Isom}(X_{\mathcal{S}_n})$. Then $Ue_i = \pm e_i$ for each $i \in \mathbb{N}$*

All the work in the section is related to the proof of the Theorem 42. Let us fix $n \in \mathbb{N}$, the isometry U and the following notation throughout this section: Let $Ue_i = x_i$ and $Uy_i = e_i$.

Remark 43. We mention two facts about a maximal set in $\mathcal{S}_n^{MAX}$ $(n \geq 1)$.

1. A set $E \in \mathcal{S}_n^{MAX}$ if and only if for each m, k with $m + k = n$ there is a unique sequence $(E_i)_{i=1}^d$ so that $E = \cup_{i=1}^d E_i$ with $(\min E_i)_{i=1}^d \in \mathcal{S}_m^{MAX}$, $E_1 < E_2 < \dots E_d$ are in $\mathcal{S}_k^{MAX}$.
2. Let $n \in \mathbb{N}$ with $m + k = n$. If a set $G \in \mathcal{S}_n^{MAX}$ is written as $\cup_{i=0}^d G_i$, where $G_0 < G_1 < \dots < G_d \in \mathcal{S}_m^{MAX}$, then $(\min G_i)_{i=0}^d \in \mathcal{S}_k^{MAX}$.

Remark 44. Suppose that $G \in \mathcal{S}_n^{MAX}$ and $F \subset \mathbb{N}$ with $\min G < \min F$, F a spread of G with $|F| = |G|$. Then if $j > \min G$, $\{j\} \cup F \in \mathcal{S}_n$.

Proof. By Remark 43 item 1, we write $G = \cup_{i=1}^d G_i$ so that $G_1 < \dots < G_d$ in $\mathcal{S}_{n-1}^{MAX}$, $(\min G_i)_{i=1}^d \in \mathcal{S}_1^{MAX}$, and $d = \min G_1$. Since $|F| = |G|$ and F is a spread of G there is a corresponding decomposition $F = \cup_{i=1}^d F_i$ where F_i is a spread of G_i. Let $j > \min G$. Then

$$\{\{j\}, F_1, \dots, F_d\}$$

is a collection of $d + 1$-many $\mathcal{S}_{n-1}$ sets and the overall minimum is greater than or equal to $d + 1$. Therefore $\{j\} \cup F \in \mathcal{S}_n$, as desired. $\square$

We require the following technical lemma.

Lemma 45. *The following hold:*

1. *We have $Ue_1 = \pm e_1$.*

2. *Let $j \in \mathbb{N}$ with $j \geqslant 2$. Then, $x_j \in c_{00}$, $x_j(1) = 0$, and x_j has a non-maximal one set.*

3. *Let $m \in \mathbb{N}$ and $j > \max\{\max\operatorname{supp} x_i : 1 \leqslant i \leqslant m\}$. Then $\min\operatorname{supp} y_j > m$.*

Proof. We prove item 1. If $\max\operatorname{supp} x_1 = 1$, we are done. Suppose there exists $k \geq 2$ such that $x_1(k) \neq 0$. Because $x_1 \in B(X_{\mathcal{S}_n})$, there exists $n \in \mathbb{N}$ with $2|x_1(n)| < |x_1(k)|$. Consider $t_n e_n$, where $t_n = \begin{cases} 1 - x_1(n) \text{ if } x_1(n) \geq 0 \\ -1 - x_1(n) \text{ if } x_1(n) < 0 \end{cases}$. Let $Uz = t_n e_n$. By definition of isometry, we have:

$$\begin{aligned} ||e_1 \pm z|| &= ||x_1 \pm t_n e_n|| \geq |x_1(k)| + |x_1(n) \pm t_n| \\ &\geq |x_1(k)| + |t_n| - |x_1(n)| \geq |x_1(k)| + (1 - |x_1(n)|) - |x_1(n)| \\ &= 1 + (|x_1(k)| - 2|x_1(n)|) > 1. \end{aligned}$$

Hence, if $F^+ \in \mathcal{S}_n$ with $\sum_{i \in F^+} |(e_1 + z)(i)| > 1$, then $1 \in F^+$. If $F^- \in \mathcal{S}_n$ with $\sum_{i \in F^-} |(e_1 - z)(i)| > 1$, then $1 \in F^-$. Therefore, $F^+ = F^- = \{1\}$ and so,

$$|1 + z(1)| = |1 - z(1)| > 1.$$

Because $-1 \leq z(1) \leq 1$, we have a contradiction. So, $\max\operatorname{supp} x_1 = 1$ or $Ue_1 = \pm e_1$.

We proceed to prove item 2. It is easy to show that $\pm e_1 + e_j \in E(X_{\mathcal{S}_n})$ for all $j \geq 2$. By Proposition 12, $U(\pm e_1 + e_j) \in E(X_{\mathcal{S}_n})$, and by Proposition 40, $U(\pm e_1 + e_j)$ has a non-maximal 1-set. So, $\pm e_1 + x_j$ has a non-maximal 1-set. This shows that $x_j \in c_{00}$ and x_j has a non-maximal 1-set. If $x_j(1) \neq 0$, then either $|e_1 + x_j(1)| > 1$ or $|e_1 - x_j(1)| > 1$, a contradiction. So, $x_j(1) = 0$.

Finally, we prove item 3 by induction. Base case: for $m = 1$, we have $\max\{\max\operatorname{supp} x_i : 1 \leq i \leq 1\} = \max\operatorname{supp} e_1 = 1$. Pick $j > 1$. We want to show that $\min\operatorname{supp} y_j > 1$. We have:

$$||e_1 \pm y_j|| = ||U(e_1 \pm y_j)|| = ||Ue_1 \pm Uy_j|| = || \pm e_1 \pm e_j|| = 1.$$

This only happens if $|1 + y_j(1)| \leq 1$ and $|1 - y_j(1)| \leq 1$, which in turn implies that $y_j(1) = 0$. So, $\min\operatorname{supp} y_j > 1$. Suppose that the statement holds true for $m \leq k$ for some $k \geq 1$. We want to show that the statement holds for $m = k + 1$. Pick $j > \max\{\max\operatorname{supp} x_i : 1 \leq i \leq k + 1\}$. Because $j > \max\{\max\operatorname{supp} x_i : 1 \leq i \leq k\}$, by our inductive hypothesis, $\min\operatorname{supp} y_j > k$. Hence, it suffices to prove that $y_j(k + 1) = 0$. By item 2, x_{k+1} has a non-maximal 1-set F. Therefore, $F \cup \{j\} \in \mathcal{S}_n$ and so, $2 = ||x_{k+1} \pm e_j||$. Therefore, $||e_{k+1} \pm y_j|| = 2$. Let $F^+ \in \mathcal{S}_n$ with $\sum_{i \in F^+} |(e_{k+1} + y_j)(i)| = 2$ and $F^- \in \mathcal{S}_n$ with

$\sum_{i\in F^-} |(e_{k+1} - y_j)(i)| = 2$. Since the norm of both of these vectors is 2, we know that $k+1 \in F^+ \cap F^-$. Therefore,

$$\begin{aligned} 2 &\geq |1 + y_j(k+1)| + \sum_{i\in F^+\setminus\{k+1\}} |y_j(i)| \\ 2 &\geq |1 - y_j(k+1)| + \sum_{i\in F^-\setminus\{k+1\}} |y_j(i)|. \end{aligned}$$

If $y_j(k+1) \neq 0$, then either $|1 + y_j(k+1)| < 1$ or $|1 - y_j(k+1)| < 1$. So, either $\sum_{i\in F^+} |y_j(i)|$ or $\sum_{i\in F^-} |y_j(i)| > 1$, which contradicts the fact that $||y_j|| = 1$. Therefore, $\min\operatorname{supp} y_j > k+1$, as desired. □

For $x, y \in c_{00}$ we write $x < y$ if $\max\operatorname{supp} x < \min\operatorname{supp} y$ and $k < x$ if $k \leqslant \min\operatorname{supp} x$. If $F \subset \mathbb{N}$ we will say that $(z_i)_{i\in F}$ is a block sequence if for $i < j$ in F $z_i < z_j$.

Corollary 46. *For each $m \in \mathbb{N}$ there is an $d \in \mathbb{N}$ and $m < y_d$ and $k \in \mathbb{N}$ with $y_d < y_k$.*

Proof. Fix $m \in \mathbb{N}$. Using Lemma 45 item 3 we can find d sufficiently large so that $m < y_d$. Applying Lemma 45 item 3 for $\max\operatorname{supp} y_d$ we can find k with $y_d < y_k$. □

Proof of Theorem 42. Fix $k \in \mathbb{N}$. We will prove that $x_k = \pm e_k$. The proof proceeds by induction. Base case: for $k = 1$, we have $x_1 = \pm e_1$ due to Lemma 45 item 1. Now fix a $k \geq 2$ and assume that the claim holds for all $i < k$. Let $k_0 = \max\{k, \max\operatorname{supp} x_k\}$. By repeated applications of Corollary 46, we can find a set $F_1 \subset \mathbb{N}$ so that $k_0 < F_1$, $|F_1| = k_0$, and a block sequence $(y_i)_{i\in F_1}$ with $\max F_1 < \sum_{i\in F_1} y_i =: z_1$.

Let $k_1 = \max\operatorname{supp} z_1$. Find $F_2 \subset \mathbb{N}$ so that $|F_2| = k_1$, $k_1 < F_2$, and a block sequence $(y_i)_{i\in F_2}$ with $\max F_2 < \sum_{i\in F_2} y_i =: z_2$.

Continuing in this way we can construct an increasing sequence $(k_i)_{i=0}^{\infty}$ so that for each i

$$z_{i+1} \;=\; \sum_{j\in F_{i+1}} y_j \;>\; \max F_{i+1}$$

with $|F_{i+1}| = k_i$ and a block sequence $(y_j)_{j\in F_{i+1}}$.

There is a unique $d(n-1) \in \mathbb{N} \cup \{0\}$ so that $(k_i)_{i=0}^{d(n-1)} \in \mathcal{S}_{n-1}^{MAX}$ (clearly, $d(0) = 0$ and $d(1) = k_0 - 1$). Consider the following two remarks.

Remark 47. Let $j > k_0$ and $F := \cup_{i=1}^{d(n-1)+1} F_i$. We claim that

$$\{j\} \cup F \in \mathcal{S}_n. \tag{1}$$

Our tool is Remark 44. Let $G_i = \{k_i, \dots, 2k_i - 1\}$ for $i \in \mathbb{N} \cup \{0\}$. Then $G_0 < G_1 < \cdots < G_{d(n-1)}$ are in $\mathcal{S}_1^{MAX}$ and $G := \cup_{i=0}^{d(n-1)} G_i \in \mathcal{S}_n^{MAX}$ by the definition of $d(n-1)$.

Note that $|F_i| = |G_{i-1}| = k_{i-1}$ (i.e. $|F| = |G|$), F is a spread of G, and $\min G = k_0 < \min F$. Therefore we can apply Remark 44 to conclude that (1) holds.

Remark 48. Suppose $G \in \mathcal{S}_n^{MAX}$ has the property that there are sets $G_0 < \cdots < G_m$ are in $\mathcal{S}_1^{MAX}$ such that $\min G_i \leqslant k_i$ with $G = \cup_{i=0}^m G_i$. Then $m \leqslant d(n-1)$. Indeed suppose $m > d(n-1)$. Since $(k_i)_{i=0}^{d(n-1)} \in \mathcal{S}_{n-1}^{MAX}$ we know that $(k_i)_{i=0}^m \notin \mathcal{S}_{n-1}$. Since $\min G_i \leqslant k_i$ we can conclude that $(\min G_i)_{i=0}^m \notin \mathcal{S}_{n-1}$. Therefore using Remark 43 item 2 we conclude that $G \notin \mathcal{S}_n^{MAX}$.

Note that by definition

$$U(e_k + \sum_{i=1}^{d(n-1)+1} \sum_{j \in F_i} y_j) = x_k + \sum_{i=1}^{d(n-1)+1} \sum_{j \in F_i} e_j.$$

We will show that if $\max \operatorname{supp} x_k \geqslant k+1$ then we have the contradiction:

1. $\|x_k + \sum_{i=1}^{d(n-1)+1} \sum_{j \in F_i} e_j\| > \sum_{i=1}^{d(n-1)+1} |F_i|$

2. $\|e_k + \sum_{i=1}^{d(n-1)+1} \sum_{j \in F_i} y_j\| \leqslant \sum_{i=1}^{d(n-1)+1} |F_i|$

First we will prove item (1).

Let $j \in \operatorname{supp} x_k$ with $j \geqslant k+1$. Using Remark 47,

$$F = \{j\} \cup \bigcup_{i=1}^{d(n-1)+1} F_i \in \mathcal{S}_n.$$

We may therefore conclude that

$$\|x_k + \sum_{i=1}^{d+1} \sum_{j \in F_i} e_j\| \geqslant |x_k(j)| + \sum_{i=1}^{d(n-1)+1} |F_i|.$$

This prove the first item.

We will now prove the second item. Fix a $G \in \mathcal{S}_n^{MAX}$ (we may assume without loss of generality that G is maximal). Then $G = \cup_{i=0}^m G_i$ where $G_0 < \cdots < G_m$ are in $\mathcal{S}_1^{MAX}$ and $(\min G_i)_{i=0}^m \in \mathcal{S}_{n-1}^{MAX}$.

First note that if either $k_0 \notin G$ or $G \cap \operatorname{supp} y_j = \emptyset$ for some $j \in \cup_{i=1}^{d(n-1)+1} F_i$ the desired upper bound follows from counting the vectors whose intersection is non-empty. Note that in total there are $1 + \sum_{i=1}^{d(n-1)+1} |F_i|$ many vectors and so missing any single vector (which, notably, have norm 1) yields the desired upper bound.

Therefore we may assume that

$$k_0 \in G \text{ and } G \cap \operatorname{supp} y_j \neq \emptyset \text{ for all } j \in \bigcup_{i=1}^{d(n-1)+1} F_i. \tag{2}$$

Therefore $k_0 \in G$ and, in particular, $\min G_0 \leqslant k_0$. Since $G_0 \in \mathcal{S}_1^{MAX}$, $k_0 < F_1$ and $|F_1| = k_0$, $G_0 \cap \operatorname{supp} y_{\max F_1} = \emptyset$. Consequently, $\min G_1 \leqslant$

$\max \operatorname{supp} y_{\max F_1} = k_1$. Continuing in this manner we see that $\min G_i \leqslant k_i$ and $G_i \cap \operatorname{supp} y_{\max F_{i+1}} = \emptyset$ for each $0 \leqslant i \leqslant m$. Therefore by Remark 48 we may conclude that $m \leqslant d(n-1)$. However,

$$G_m \cap \operatorname{supp} y_{\max F_{m+1}} = \emptyset$$

and $m \leqslant d(n-1)$ contradicts (2) and yields the desired upper bound.

Therefore we can conclude, as desired, that $\max \operatorname{supp} x_k \leqslant k$. By induction, we know that $Ue_j = \varepsilon_j e_j$ for each $j < k$. If $k = 2$ we have from Lemma 45 item 1 that $x_k(1) = 0$ and thus $x_k = \pm e_k$. Suppose $k \geqslant 3$ and let $j < k$. If $j = 1$, $x_k(j) = 0$ by Lemma 45 item 2. Suppose then that $1 < j < k$. Then

$$2 \;=\; \|e_j \pm e_k\| \;=\; \|\varepsilon_j e_j \pm x_k\|$$

Arguing as in the proof of Lemma 45 item 3, we know that if $\sum_{i \in F^+} |(\varepsilon_j e_j + x_k)(i)| = 2$ for $F^+ \in \mathcal{S}_n$ then $j \in F^+$ and if $\sum_{i \in F^-} |(\varepsilon_j e_j - x_k)(i)| = 2$ for $F^- \in \mathcal{S}_n$ then $j \in F^-$. Therefore

$$2 \;=\; |\varepsilon_j + x_k(j)| + \sum_{i \in F^+, i \neq j} |x_k(i)|,$$

$$2 \;=\; |\varepsilon_j - x_k(j)| + \sum_{i \in F^-, i \neq j} |x_k(i)|.$$

Consequently, if $x_k(j) \neq 0$ we can see that either $\sum_{\{i \in F^+, i \neq j\}} |x_k(i)|$ or $\sum_{\{i \in F^+, i \neq j\}} |x_k(i)|$ is strictly greater than 1. This contradicts the fact that $\|x_k\| \leqslant 1$.

Whence $\operatorname{supp} x_k = \{k\}$. Since x_k is a norm one vector $x_k = \pm e_k$ which is the desired result. $\square$

Part IV

BONUS RESULT

8

LINEAR RECURRENCE RELATION FROM THE GENERALIZED SCHREIER SETS

Fibonacci numbers have been discovered to hide under many different forms in mathematics. An online post [13] on a website devoted to the Banach space theory proves that the Fibonacci sequence appears if we count the Schreier sets under a certain condition. In particular, define $M_{1,n} = \{S \in \mathcal{S}_1 : \max S = n\}$. Then $|M_{1,1}| = 1$, $|M_{1,2}| = 1$ and $|M_{1,n+2}| = |M_{1,n+1}| + |M_{1,n}|$ for all $n \geq 1$. We first show two proofs of the below theorem.

Theorem 49. *The sequence $(|M_{1,n}|)_{n=1}^{\infty}$ is the Fibonacci sequence.*

The first proof ([13]) is very elegant. It uses two one-to-one mappings to argue about an equality of cardinalities of sets. The second proof is more computational and can be easily extended to prove the general case. We generalize Theorem 49 as follows: define $\mathcal{S}_m = \{S \subseteq \mathbb{N} : \lfloor \min S/m \rfloor \geq |S|\}$, $M_{m,n} = \{S \in \mathcal{S}_m : \max S = n\}$, and prove the following theorem.

Theorem 50. *Given $m \in \mathbb{N}$, consider the sequence $(|M_{m,n}|)_{n=1}^{\infty}$. We have:*

1. *For $n \leq m-1$, $|M_{m,n}| = 0$,*
2. *For $m \leq n \leq m+1$, $|M_{m,n}| = 1$,*
3. *For $n \geq m+2$, $|M_{m,n}| = |M_{m,n-1}| + |M_{m,n-1-m}|$.*

We call $(|M_{m,n}|)_{n=1}^{\infty}$ the generalized Fibonacci sequence of order m.

8.1 TWO PROOFS OF THEOREM 49

Given a set A of natural numbers, define $A \pm 1 = \{a | a \pm 1 \in A\}$.

First proof of Theorem 49 from [13].
Because $|M_n| = |M_{n+1}| = 1$. It suffices to prove that $|M_n| + |M_{n+1}| = |M_{n+2}|$ for all $n \geq 1$.

Given a Schreier set S, define $R_n(S) = S \cup \{n\} \setminus \{\max S\}$ and $T_n(S) = (S+1) \cup \{n\}$. In words, R_n replaces the maximum of S with n; T_n increases each element of S by 1 and add element n to the set.

Let $X \in M_{n+1}$ be chosen. We have $R_{n+2}(X) \in M_{n+2}$ because $\max R_{n+2}(X) = n+2$ and $R_{n+2}(X) \in \mathcal{S}_1$. We can write $R_{n+2} : M_{n+1} \to M_{n+2}$ and see that R_{n+2} is one-to-one, and given a set $V \in M_{n+2}$ with $n+1 \notin V$, we can find a set U in M_{n+1} such that $R_{n+2}(U) = V$. Particularly, $U = V \cup \{n+1\} \backslash \{n+2\}$. Therefore, $|M_{n+1}| = |R_{n+2}(M_{n+1})| = |\{S \in M_{n+2} : (n+1) \notin S\}|$.

Let $X \in M_n$ be chosen. We have $T_{n+2}(X) \in M_{n+2}$ since $\max T_{n+2}(X) = n+2$ and $\min T_{n+2}(X) = \min X + 1$, making $T_{n+2}(X) \in \mathcal{S}_1$ though $|T_{n+2}(X)| = |X| + 1$. We can write $T_{n+2} : M_n \to M_{n+2}$ and see that T_{n+2} is one-to-one. Given a set $V \in M_{n+2}$ with $n+1 \in V$, we can find a set U in M_n such that $T_{n+2}(U) = V$. Particularly, $U = V \backslash \{n+2\} - 1$. Therefore, $|M_n| = |T_{n+2}(X)| = |\{S \in M_{n+2} : (n+1) \in S\}|$.

Therefore,

$$\begin{aligned} |M_n| + |M_{n+1}| &= |\{S \in M_{n+2} : (n+1) \notin S\}| \\ &\quad + |\{S \in M_{n+2} : (n+1) \in S\}| = |M_{n+2}|. \end{aligned} \tag{3}$$

Due to Equation 3, we complete the proof. □

Second proof of Theorem 49.
Given $n \in \mathbb{N}$, we split:

$$\begin{aligned} M_{1,n} &= \{S \in \mathcal{S}_1 : \min S = 1, \max S = n\} + \\ &\quad \{S \in \mathcal{S}_1 : \min S = 2, \max S = n\} + \\ &\quad \{S \in \mathcal{S}_1 : \min S = 3, \max S = n\} + \\ &\quad \ldots + \{S \in \mathcal{S}_1 : \min S = n = \max S = n\} \\ &= \cup_{k=1}^{n} \{S \in \mathcal{S}_1 : \min S = k, \max S = n\}. \end{aligned}$$

We define $\binom{m}{n} = 0$ if $n > m$ or $m < 0$ and write:

$$|\{S \in \mathcal{S}_1 : \min S = k, \max S = n\}| = \begin{cases} \sum_{j=0}^{k-2} \binom{n-k-1}{j} \text{ if } k < n \\ 1 \text{ if } k = n \end{cases},$$

where j is the possible number of elements added to a Schreier set with minimum k and maximum n. Therefore, we have:

$$M_{1,n} = \sum_{k=1}^{n-1} \sum_{j=0}^{k-2} \binom{n-k-1}{j} + 1 = \sum_{k=2}^{n-1} \sum_{j=0}^{k-2} \binom{n-(k+1)}{j} + 1.$$

It can be verified that $M_{1,1} = M_{1,2} = 1$, and so, it suffices to prove that $M_{1,n} + M_{1,n+1} = M_{1,n+2}$ for all $n \geq 1$. We have:

$$\begin{aligned} M_{1,n+2} - M_{1,n+1} &= \sum_{k=1}^{n+1} \sum_{j=0}^{k-2} \binom{n-k+1}{j} - \sum_{k=1}^{n} \sum_{j=0}^{k-2} \binom{n-k}{j} \\ &= \sum_{k=1}^{n} \sum_{j=0}^{k-2} \left(\binom{n-k+1}{j} - \binom{n-k}{j} \right) + 1 \\ &= \sum_{k=1}^{n} \sum_{j=0}^{k-2} \binom{n-k}{j-1} + 1 = \sum_{k=3}^{n} \sum_{j=0}^{k-2} \binom{n-k}{j-1} + 1. \end{aligned}$$

To prove that $M_{1,n+2} - M_{1,n+1} = M_{1,n}$, we want to show that:

$$\sum_{k=3}^{n}\sum_{j=0}^{k-2}\binom{n-k}{j-1} = \sum_{k=2}^{n-1}\sum_{j=0}^{k-2}\binom{n-(k+1)}{j}. \tag{4}$$

Pick $t \in \mathbb{N}_0$ such that $t+2 \le n-1$. Equation 4 is true if we can show:

$$\sum_{j=0}^{(3+t)-2}\binom{n-(3+t)}{j-1} = \sum_{j=0}^{(2+t)-2}\binom{n-((2+t)+1)}{j}.$$

Equivalently,

$$\sum_{j=0}^{t+1}\binom{n-t-3}{j-1} = \sum_{j=0}^{t}\binom{n-t-3}{j},$$

which is true. This completes our proof. □

8.2 PROOF OF THEOREM 50

The proof of Theorem 50 is simply a generalization of the second proof of Theorem 49. Therefore, we present only the key components of the proof. Similar to the case where $m=1$ above,

$$|M_{m,n}| = \sum_{k=1}^{n-1}\sum_{j=0}^{\lfloor k/m\rfloor-2}\binom{n-k-1}{j} + \begin{cases}1 \text{ if } \lfloor n/m\rfloor \ge 1\\ 0 \text{ if } \lfloor n/m\rfloor < 1\end{cases}.$$

We want to show that for $n \ge m+2$, $|M_{m,n}| = |M_{m,n-1}| + |M_{m,n-m-1}|$ by proving the following lemma.

Lemma 51. *Fix $m \ge 1$, for $n \ge m+2$, we have:*

$$\sum_{k=1}^{n-1}\sum_{j=0}^{\lfloor k/m\rfloor-2}\binom{n-k-1}{j} = \sum_{k=1}^{n-2}\sum_{j=0}^{\lfloor k/m\rfloor-2}\binom{n-k-2}{j} + \sum_{i=1}^{n-m-2}\sum_{j=0}^{\lfloor k/m\rfloor-2}\binom{n-m-k-2}{j} + g(n),$$

where $g(n) = \begin{cases}0 \text{ if } \lfloor (n-m-1)/m\rfloor \ge 1\\ 1 \text{ if } \lfloor (n-m-1)/m\rfloor < 1\end{cases}.$

Proof. We have:

$$\begin{aligned}&\sum_{k=1}^{n-1}\sum_{j=0}^{\lfloor k/m\rfloor-2}\binom{n-k-1}{j} - \sum_{k=1}^{n-2}\sum_{j=0}^{\lfloor k/m\rfloor-2}\binom{n-k-2}{j}\\ &= \sum_{k=1}^{n-2}\sum_{j=0}^{\lfloor k/m\rfloor-2}\left[\binom{n-k-1}{j} - \binom{n-k-2}{j}\right] + \sum_{j=0}^{\lfloor (n-1)/m\rfloor-2}\binom{0}{j}\\ &= \sum_{k=1}^{n-2}\sum_{j=0}^{\lfloor k/m\rfloor-2}\binom{n-k-2}{j-1} + \sum_{j=0}^{\lfloor (n-1)/m\rfloor-2}\binom{0}{j}.\end{aligned}$$

Because $\sum_{j=0}^{\lfloor (n-1)/m \rfloor -2} \binom{0}{j} = g(n)$, it suffices to prove that

$$\sum_{k=1}^{n-2} \sum_{j=0}^{\lfloor k/m \rfloor -2} \binom{n-k-2}{j-1} = \sum_{k=1}^{n-m-2} \sum_{j=0}^{\lfloor k/m \rfloor -2} \binom{n-m-k-2}{j}.$$

Equivalently,

$$\sum_{k=3m}^{n-2} \sum_{j=0}^{\lfloor k/m \rfloor -2} \binom{n-k-2}{j-1} = \sum_{i=2m}^{n-m-2} \sum_{j=0}^{\lfloor k/m \rfloor -2} \binom{n-m-k-2}{j}. \tag{5}$$

Fix $t \in \mathbb{N}_0$ such that $3m + t \leq n - 2$. Equality 5 is proved if we can show that

$$\sum_{j=0}^{\lfloor (3m+t)/m \rfloor -2} \binom{n-(3m+t)-2}{j-1} = \sum_{j=0}^{\lfloor (2m+t)/m \rfloor -2} \binom{n-m-(2m+t)-2}{j}.$$

Equivalently,

$$\sum_{j=0}^{\lfloor t/m \rfloor +1} \binom{n-3m-t-2}{j-1} = \sum_{j=0}^{\lfloor t/m \rfloor} \binom{n-3m-t-2}{j},$$

which is true. We have completed the proof. □

Corollary 52. *Given $m \in \mathbb{N}$, for $n \geq m+2$, $|M_{m,n}| = |M_{m,n-1}| + |M_{m,n-m-1}|$.*

Given $m \in \mathbb{N}$, we consider the sequence $(|M_{m,n}|)_{n=1}^{\infty}$.

1. For $1 \leq n \leq m-1$, $|M_{m,n}| = |\{S \in \mathcal{S}_m | \max S = n \leq m-1\}|$. Because for all $S \in \mathcal{S}_m$, $\min S \geq m$, $|M_{m,n}| = 0$,
2. For $n = m$, $|M_{m,m}| = |\{S \in \mathcal{S}_m | \max S = m\}|$. Hence, if $X \in M_{m,m}$, $\min X = \max X = m$ or $X = \{m\}$. So, $|M_{m,m}| = 1$,
3. For $n = m+1$, $M_{m,m+1} = |\{S \in \mathcal{S}_m | \max S = m+1\}|$. Hence, if $X \in M_{m,m+1}$, $\min X \in \{m, m+1\}$ and $\max X = m+1$. If $\min X = m$, then $|X| = 1$ and so, X cannot contain $m+1$, a contradiction. So, $X = \{m+1\}$, and $|M_{m,m+1}| = 1$.

We see that the first $m-1$ numbers of the sequence $(|M_{m,n}|)_{n=1}^{\infty}$ are zero, while the next two are 1. Also, for all $n \geq m+2$, $|M_{m,n}| = |M_{m,n-1}| + |M_{m,n-m-1}|$ by Lemma 51. We have shown that $(|M_{m,n}|)_{n=1}^{\infty}$ is a higher-order Fibonacci sequence.

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