Compressions of Linear Operators Yielding a Single Point Numerical Range

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1. Introduction

Given a linear operator $T : V \to V$ on some inner-product space $(V, \langle ., . \rangle)$, we define the numerical range of T as

$$W(T) = \{\langle Tv, v\rangle : v \in V, \|v\| = 1\}.$$

One should note that unlike the range of T, which is a collection of vectors in V, the numerical range of T is a collection of scalars. The numerical range has a number of interesting properties; for example, $W(T)$ must be convex, and if T is a normal operator and V is finite dimensional, $W(T)$ is the convex hull of the eigenvalues of T.

This thesis will focus on finding subspaces M such that the compression of T to M, denoted T_M, has a single point numerical range. We define $T_M : M \to M$ by

$$T_M v = P_M T v, \forall v \in M$$

where P_M is the the orthogonal projection of V onto M.

The motivation for finding such subspaces lies in a theorem from quantum coding theory, which states that an error process is correctable on a subspace M if the compression to M of each member of a particular collection of linear operators associated with that error yields a single point numerical range. It is, in particular, desireable to find such subspaces M of highest possible dimension.

If $W(T_M) = \{\alpha\}$, it can easily be shown that α is in the numerical range of T, and this raises the question of which points z in the numerical range of T have associated subspaces M_z such that $W(T_{M_z}) = \{z\}$ where M_z is of a given dimension. It is this question which will be the main focus of this thesis. We will consider subsets $W_r(T)$ (where $r = 1, 2, \dots$) of $W(T)$ defined by

$$W_r(T) = \{\alpha \in W(T) : \{\alpha\} = W(T_M) \text{ where } \dim(M) = r\}$$

and investigate their properties. In addition we will arrive at a result completely characterizing the sets $W_r(T)$ for any self-adjoint operator T.

We will conclude with an investigation of $W_r(T)$ for T a normal operator, finding subsets of the numerical range of T in which $W_r(T)$ must be contained, given r an integer such that $W_r(T)$ is non-empty, and establishing a complete characterization of the subsets $W_r(T)$ of $W(T)$ when $T : \mathbb{C}^n \to \mathbb{C}^n$ has distinct eigenvalues forming a convex n-gon and $n \leq 6$. We will finally conjecture that for T normal, $W_r(T)$ is a convex set for every r.

2. Backround

The results to follow require several basic concepts from linear algebra, beginning with that of an *inner product*:

Definition 2.1. Let V be a vector space over $\mathbb{C}$ and let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$. Then $\langle \cdot, \cdot \rangle$ is an *inner product*, and $(V, \langle \cdot, \cdot \rangle)$ is an *inner-product space* if $\langle \cdot, \cdot \rangle$ satisfies:

(a) $\forall v \in V, \langle v, v\rangle \geq 0$;
(b) $\langle v, v\rangle = 0$ if and only if $v = 0$;
(c) $\forall v, w, u \in V, a \in \mathbb{C}, \langle av + u, w\rangle = a\langle v, w\rangle + \langle u, w\rangle$;
(d) $\forall v, w \in V, \langle v, w\rangle = \overline{\langle w, v\rangle}$.

We quickly derive several additional useful properties of the inner product.

Proposition 2.2. *Suppose that $\langle \cdot, \cdot \rangle$ is an inner product on V, that $v, w, u \in V$, and that $a \in \mathbb{C}$. Then*

(a) $\langle v, aw\rangle = \bar{a}\langle v, w\rangle$;
(b) $\langle v, w + u\rangle = \langle v, w\rangle + \langle v, u\rangle$;
(c) $\langle v, 0\rangle = \langle 0, v\rangle = 0$.

Proof. To show (a), we observe that

$$\begin{aligned} \langle v, aw\rangle &= \overline{\langle aw, v\rangle} \\ &= \overline{a\langle w, v\rangle} \\ &= \bar{a}\overline{\overline{\langle v, w\rangle}} \\ &= \bar{a}\langle v, w\rangle. \end{aligned}$$

It is a similar task to show (b):

$$\begin{aligned}
\langle v, w+u\rangle &= \overline{\langle w+u, v\rangle} \\
&= \overline{\langle w, v\rangle + \langle u, v\rangle} \\
&= \overline{\langle w, v\rangle} + \overline{\langle u, v\rangle} \\
&= \langle v, w\rangle + \langle v, u\rangle .
\end{aligned}$$

To show (c), we observe that

$$\begin{aligned}
\langle v, 0\rangle &= \langle v, 0v\rangle \\
&= \overline{0}\langle v, v\rangle \\
&= 0 \\
&= 0\langle v, v\rangle \\
&= \langle 0v, v\rangle \\
&= \langle 0, v\rangle .
\end{aligned}$$

□

One common example of an inner product is that on $\mathbb{C}^n$, given by

$$\left\langle \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{bmatrix} \right\rangle = \sum_{i=1}^{n} v_i \overline{w_i}.$$

One useful property of the inner product is that it gives rise to a norm.

Definition 2.3. A *norm* on a vector space V is a function $\|\cdot\|$ from V to $\mathbf{R}$ satisfying the following properties:
(a) $\|v\| \geq 0 \ \forall v \in V$;
(b) $\|v\| = 0$ if and only if $v = 0$;
(c) $\|av\| = |a|\|v\| \ \forall v \in V, a \in \mathbb{C}$;
(d) $\|v+w\| \leq \|v\| + \|w\| \ \forall v, w \in V$ (the triangle inequality).

We may now define the norm of $v \in V$ for V an inner-product space to be $\|v\| = \sqrt{\langle v, v\rangle}$. Note that by the definition of the inner product this is in fact a function from V to R, as $\langle v, v\rangle \geq 0$. To verify that this satisfies all the necessary properties of a norm, we will first require the following important inequality.

Theorem 2.4 (Cauchy-Schwartz Inequality). *Suppose V is an inner-product space and $w, v \in V$. Then $|\langle w, v\rangle| \leq \|w\|\|v\|$. Furthermore, given v nonzero, this will be an equality if and only if w is a scalar multiple of v.*

Proof. If v is 0, the inequality is clearly satisfied. Now suppose $v \neq 0$, so $\langle v, v\rangle \neq 0$, and thus $\|v\| \neq 0$. Then

$$\begin{aligned}
0 &\leq \|w - \frac{\langle w, v\rangle}{\|v\|^2} v\|^2 \\
&= \left\langle w - \frac{\langle w, v\rangle}{\|v\|^2} v, w - \frac{\langle w, v\rangle}{\|v\|^2} v \right\rangle \\
&= \langle w, w\rangle - \frac{\langle w, v\rangle}{\|v\|^2}\langle v, w\rangle - \frac{\overline{\langle w, v\rangle}}{\|v^2\|}\langle w, v\rangle + \frac{|\langle w, v\rangle|^2}{\|v\|^4}\langle v, v\rangle \\
&= \|w\|^2 - \frac{|\langle w, v\rangle|^2}{\|v\|^2} - \frac{|\langle w, v\rangle|^2}{\|v\|^2} + \frac{|\langle w, v\rangle|^2}{\|v\|^2} \\
&= \|w\|^2 - \frac{|\langle w, v\rangle|^2}{\|v\|^2}.
\end{aligned}$$

Hence, $\frac{|\langle w,v\rangle|^2}{\|v\|^2} \leq \|w\|^2$, and thus $|\langle w, v\rangle|^2 \leq \|w\|^2\|v\|^2$, so that $|\langle w, v\rangle| \leq \|w\|\|v\|$. Now observe that we have equality in the equation only if $\left\langle w - \frac{\langle w,v\rangle}{\|v\|^2} v, w - \frac{\langle w,v\rangle}{\|v\|^2} v \right\rangle = 0$, so that $w - \frac{\langle w,v\rangle}{\|v\|^2} v = 0$, and thus $w = \frac{\langle w,v\rangle}{\|v\|^2} v$, so that

w is a scalar multiple of v. Finally, suppose $w = av$, where $a \in \mathbb{C}$. Then

$$\begin{aligned}
\|w\|\|v\| &= \sqrt{\langle av, av\rangle}\sqrt{\langle v, v\rangle} \\
&= |a| \langle v, v\rangle \\
&= |a \langle v, v\rangle | \\
&= | \langle av, v\rangle | \\
&= | \langle w, v\rangle |
\end{aligned}$$

so that equality is guaranteed for w a scalar multiple of v, completing the proof. □

Proposition 2.5. *The function from an inner-product space V to $\mathbf{R}$ given by $\|v\| = \sqrt{\langle v, v\rangle}$ is a norm on V.*

Proof. Properties (a), (b) and (c) follow directly from the definition of an inner product. It remains to show property (d), the triangle inequality, which will follow directly from the Cauchy-Schwartz inequality. Let $v, w \in V$. Then

$$\begin{aligned}
\|v + w\|^2 &= \langle v + w, v + w\rangle \\
&= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\
&\leq \|v\|^2 + \|v\|\|w\| + \|w\|\|v\| + \|w\|^2 \\
&= (\|v\| + \|w\|)^2
\end{aligned}$$

and hence $\|v + w\| \leq \|v\| + \|w\|$. □

We now provide some additional definitions concerning inner-product spaces.

Definition 2.6. Two vectors v, w in an inner-product space V are *orthogonal* and write $v \perp w$ if $\langle v, w\rangle = 0$. A set of vectors in V is *orthonormal* if its elements are pairwise orthogonal an have norm 1.

As the introduction mentions, this thesis will primarily be concerned with certain types of linear operators. Recall that a *linear operator* on a vector space V is a function $T : V \to V$ such that $\forall v, w \in V$ and $a, b \in \mathbb{C}$, $T(ax + by) = aT(x) + bT(y)$.

Although the concepts to follow may be applied to infinite dimensional inner-product spaces, we shall for simplicity assume for the remainder of this thesis that our inner-product spaces are finite dimensional, except where noted otherwise. We may in fact then assume we are working in $\mathbb{C}^n$. Recalling that any linear operator from $\mathbb{C}^n$ to $\mathbb{C}^n$ may be represented as an $n \times n$ matrix, we will furthermore refer to linear operators and their matrix representations interchangebly.

Using the inner product we may now define two special types of linear operators that will be of special interest; first, though, we must introduce the concept of the *adjoint* of an operator.

Definition 2.7. The *adjoint* of a linear operator T on $\mathbb{C}^n$, denoted T^*, is the linear operator whose matrix representation is the conjugate transpose of that of T; that is, if $[t_{ij}]$ is the matrix representation of T relative to some given basis of $\mathbb{C}^n$, then the matrix of T^* with respect to that basis has (i, j)-entry $\overline{t_{ji}}$.

Note that it follows directly from the definition that $(T^*)^* = T$. The adjoint of an operator has an important relationship to inner products involving the operator.

Proposition 2.8. *For T a linear operator on $\mathbb{C}^n$ and $v, w \in \mathbb{C}^n$, $\langle Tv, w\rangle = \langle v, T^*w\rangle$. Furthermore, if T' is a linear operator such that $\langle Tv, w\rangle = \langle v, T'w\rangle$ for all $v, w \in \mathbb{C}^n$, then $T' = T^*$.*

Proof. Observe that if we consider vectors v and w to be n by 1 matrices, we have

$$\begin{aligned}
\langle Tv, w\rangle &= w^*Tv \\
&= (T^*w)^*v \\
&= \langle v, T^*w\rangle .
\end{aligned}$$

Now suppose $\langle Tv, w\rangle = \langle v, T'w\rangle$ for all $v, w \in \mathbb{C}^n$. Then $\langle v, T^*w\rangle = \langle v, T'w\rangle$, and thus $\langle v, (T^* - T')w\rangle = 0$. But as v is arbitrary in $\mathbb{C}^n$, we may let $v = (T^* - T')w$, so that for all $w \in \mathbb{C}^n$, $\langle (T^* - T')w, (T^* - T')w\rangle = 0$, and hence $(T^* - T')w = 0$, and thus $T^* - T'$ is the zero operator, as w is arbitrary in $\mathbb{C}$. Hence, $T' = T^*$.

□

We may now define *normal* and *self-adjoint* (or *Hermitian*) operators.

Definition 2.9. An operator T is *normal* if $TT^* = T^*T$.

Example 2.10. $T = \begin{bmatrix} 1+i & 1 \\ -1 & 1-i \end{bmatrix}$ *is normal.*

Proof. Observe that $TT^* = \begin{bmatrix} 1+i & 1 \\ -1 & 1-i \end{bmatrix}\begin{bmatrix} 1-i & -1 \\ 1 & 1+i \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, while $T^*T = \begin{bmatrix} 1-i & -1 \\ 1 & 1+i \end{bmatrix}\begin{bmatrix} 1+i & 1 \\ -1 & 1-i \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. Hence $TT^* = T^*T$. □

Definition 2.11. T is *self-adjoint* (or *Hermitian*) if $T = T^*$.

Example 2.12. $\begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ *is clearly self-adjoint.*

Observe that all Hermitian operators are clearly normal. Our investigation of these types of operators will make extensive use of the properties of their *eigenvalues* and their associated *eigenvectors.*

Definition 2.13. Let T be a linear operator on V. Then if $Tv = \lambda v$ for some $\lambda \in \mathbb{C}$ and some $v \in V \setminus \{0\}$, then λ is an *eigenvalue* of T, and v is an associated *eigenvector.*

Example 2.14. *The matrix* $\begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ *has eigenvalues* $0, 2,$ *and* 3.

Proof. Observe that $\begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Furthermore, $\begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2i \\ 2 \\ 0 \end{bmatrix}$. Finally, $\begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$. □

Notice that the matrix in the example above is the self-adjoint matrix from Example ??, and that all of its eigenvalues are real numbers, even though the matrix itself contains complex entries, as do some of the demonstrated eigenvectors. In fact, self-adjoint operators may have only real numbers as eigenvalues.

Proposition 2.15. *Suppose T is a self-adjoint operator on $\mathbb{C}^n$. Then the eigenvalues of T are real numbers.*

Proof. Let λ be an eigenvalue of T. Then $\exists v \in \mathbb{C}^n$ such that $v \neq 0$ and $Tv = \lambda v$. Now, as $\langle Tv, v\rangle = \langle v, T^*v\rangle$,

$$\begin{aligned} \langle \lambda v, v\rangle &= \langle v, T^*v\rangle \\ \langle \lambda v, v\rangle &= \langle v, Tv\rangle \\ \langle \lambda v, v\rangle &= \langle v, \lambda v\rangle \\ \lambda \langle v, v\rangle &= \overline{\lambda} \langle v, v\rangle \\ \lambda &= \overline{\lambda} \end{aligned}$$

and hence, $\lambda \in \mathbf{R}$. □

Recall the eigenvectors associated with distinct eigenvalues are linearly independent. For normal operators, even more is true; such eigenvectors must be orthogonal. Our proof will depend on the following lemma.

Lemma 2.16. *Suppose T is normal and $Tv = \lambda v$. Then $T^*v = \overline{\lambda}v$.*

Proof. Observe that

$$\begin{aligned}
\langle (T^* - \overline{\lambda} I)v, (T^* - \overline{\lambda} I)v \rangle &= \langle T^*v, T^*v \rangle - \langle T^*v, \overline{\lambda} v \rangle - \langle \overline{\lambda} v, T^*v \rangle + |\lambda|^2 \langle v, v \rangle \\
&= \langle TT^*v, v \rangle - \lambda \langle v, Tv \rangle - \overline{\lambda} \langle Tv, v \rangle + |\lambda|^2 \langle v, v \rangle \\
&= \langle T^*Tv, v \rangle - \lambda \langle v, \lambda v \rangle - \overline{\lambda} \langle \lambda v, v \rangle + |\lambda|^2 \langle v, v \rangle \\
&= \langle Tv, Tv \rangle - |\lambda|^2 \langle v, v \rangle - |\lambda|^2 \langle v, v \rangle + |\lambda|^2 \langle v, v \rangle \\
&= |\lambda|^2 \langle v, v \rangle - |\lambda|^2 \langle v, v \rangle \\
&= 0.
\end{aligned}$$

Hence, $(T^* - \overline{\lambda} I)v = 0$, so $T^*v = \overline{\lambda} v$.

□

Proposition 2.17. *Suppose T is a normal and λ_1, λ_2 are distinct eigenvalues of T. Then if v_1 and v_2 are eigenvectors associated with λ_1 and λ_2, respectively, v_1 and v_2 are orthogonal.*

Proof. Observe that

$$\begin{aligned}
\lambda_2 \langle v_1, v_2 \rangle &= \langle v_1, \overline{\lambda_2} v_2 \rangle \\
&= \langle v_1, T^* v_2 \rangle \\
&= \langle T v_1, v_2 \rangle \\
&= \langle \lambda_1 v_1, v_2 \rangle \\
&= \lambda_1 \langle v_1, v_2 \rangle .
\end{aligned}$$

Hence, $\lambda_2 \langle v_1, v_2 \rangle - \lambda_1 \langle v_1, v_2 \rangle = 0$, and thus $(\lambda_2 - \lambda_1) \langle v_1, v_2 \rangle = 0$. As λ_1, λ_2 are distinct, $(\lambda_2 - \lambda_1) \neq 0$, and thus $\langle v_1, v_2 \rangle = 0$.

□

Lemma ?? also serves as a lemma for the proof of the following theorem (for details, see [?, p. 126]).

Theorem 2.18. *Suppose $T : \mathbb{C}^n \to \mathbb{C}^n$ is a normal linear operator. Then there is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of T.*

We now arrive at the main topic of this thesis, the *numerical range.*

Definition 2.19. For M a subspace of an inner-product space V, the *unit sphere* of M, denoted $(M)_1$, is the set of all vectors in M with norm 1.

Definition 2.20. The *numerical range* of a linear operator T acting on an inner-product space V, denoted $W(T)$, is given by $W(T) = \{\langle Tv, v \rangle : v \in (V)_1\}$.

Example 2.21. *Suppose T is the identity on V. Then $W(T) = \{1\}$.*

Proof. Let $v \in (V)_1$. Then $\langle Tv, v \rangle = \langle v, v \rangle = \|v\|^2 = 1$. □

Example 2.22. *Let $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $W(T)$ is the closed disk of radius $\frac{1}{2}$ centered at the origin.*

Proof. Let $v \in (\mathbb{C}^2)_1$. Then we may write $v = \begin{bmatrix} r_1 e^{i\theta_1} \\ r_2 e^{i\theta_2} \end{bmatrix}$, where $r_1, r_2 \in [0,1], \theta_1, \theta_2 \in [0, 2\pi]$. Note that as $\|v\| = 1$, $r_1^2 + r_2^2 = 1$, so $r_1 = \sqrt{1 - r_2^2}$. Then

$$\begin{aligned}
\langle Tv, v \rangle &= \left\langle \begin{bmatrix} r_2 e^{i\theta_2} \\ 0 \end{bmatrix}, \begin{bmatrix} r_1 e^{i\theta_1} \\ r_2 e^{i\theta_2} \end{bmatrix} \right\rangle \\
&= r_1 r_2 e^{i(\theta_2 - \theta_1)}.
\end{aligned}$$

Now $\theta_2 - \theta_1$ may clearly take on any value in $[0, 2\pi]$, and $r_1 r_2 = r_2\sqrt{1 - r_2^2}$, where r_2 may be any value in $[0,1]$. It may be shown using elementary calculus that $r\sqrt{1 - r^2}$ takes $[0,1]$ surjectively to $[0, \frac{1}{2}]$. Hence, $W(M) = \{re^{i\theta} : r \in [0, \frac{1}{2}], \theta \in [0, 2\pi]\}$, the closed disk of radius $\frac{1}{2}$ centered at the origin.

□

Example 2.23. *Let T be a linear operator on V with eigenvalue λ. Then $\lambda \in W(T)$.*

Proof. Let λ be an eigenvalue of a linear operator T and let v be an associated eigenvector. Then v is non-zero, so $\|v\| \neq 0$. Then $u = \frac{v}{\|v\|}$ is a unit vector, and thus $\langle Tu, u\rangle \in W(T)$.

Now observe that

$$\begin{aligned}\langle Tu, u\rangle &= \left\langle T\frac{v}{\|v\|}, u\right\rangle \\ &= \left\langle \lambda\frac{v}{\|v\|}, u\right\rangle \\ &= \lambda\langle u, u\rangle \\ &= \lambda.\end{aligned}$$

Hence $\lambda \in W(T)$. □

One important feature of the numerical range on finite dimensional inner-product spaces is that it must be compact.

Proposition 2.24. *For T a linear operator on $\mathbb{C}^n$, $W(T)$ is compact.*

Proof. Let $f : \mathbb{C}^n \to \mathbb{C}$ be given by $f(v) = \langle Tv, v\rangle$ for all $v \in \mathbb{C}^n$. Write $v = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$, and $T = [t_{ij}]$. Then $f(v) = \sum_{l=1}^{n}\sum_{k=1}^{n} t_{lk}v_k\overline{v_l}$, which is simply a polynomial with complex coefficients in the real and imaginary parts of the v_i's, and hence is continuous. Thus, $f((\mathbb{C}^n)_1)$ is compact, as $(\mathbb{C}^n)_1$ is compact. □

(This need not be the case on infinite dimensional inner-product spaces, as the unit sphere of an inner-product space of infinite dimension need not be compact, see, for example, [?]).

One consequence of Theorem **??** is that any normal operator may be represented as a diagonal matrix (this is simply the matrix representation of the operator with respect to the orthonormal basis of eigenvectors of the operator). Using such a representation it becomes simple to determine the numerical range of the operator.

Example 2.25. *Suppose T is the normal operator with matrix representation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1+i \end{bmatrix}$. Then the numerical range of T is the filled-in triangle with vertices $1, i$, and $1+i$ (note that $1, i$, and $1+i$ are the eigenvalues of T).*

Proof. Let $v \in (\mathbb{C}^3)_1$. Then $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $|a|^2 + |b|^2 + |c|^2 = 1$. Now observe that

$$\begin{aligned}\langle Tv, v\rangle &= \left\langle \begin{bmatrix} a \\ bi \\ (1+i)c \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\rangle \\ &= |a|^2 + i|b|^2 + (1+i)|c|^2\end{aligned}$$

so that the numerical range of T consists precisely of all linear combinations of $1, i$, and $1+i$ with nonnegative coefficients summing to 1. Furthermore, note that

$$|a|^2 + i|b|^2 + (1+i)|c|^2 = |a|^2 + (|b|^2 + |c|^2)\left(i\frac{|b|^2}{|b|^2+|c|^2} + (1+i)\frac{|c|^2}{|b|^2+|c|^2}\right)$$

Notice that as $\frac{|b|^2}{|b|^2+|c|^2} + \frac{|c|^2}{|b|^2+|c|^2} = 1$, $i\frac{|b|^2}{|b|^2+|c|^2} + (1+i)\frac{|c|^2}{|b|^2+|c|^2}$ is simply some point γ on the line segment joining i and $1+i$, and so as $|a|^2 + (|b|^2+|c|^2) = 1$, $|a|^2 + (|b|^2+|c|^2)\left(i\frac{|b|^2}{|b|^2+|c|^2} + (1+i)\frac{|c|^2}{|b|^2+|c|^2}\right)$ is a point on the line segment joining γ and 1. Hence, the numerical range of T is simply the triangle with vertices $1, i$, and $1+i$ and its interior. □

In the example above we see that the numerical range of the normal operator T is in the region formed by taking all linear combinations of the eigenvalues of T such that the coefficients of the eigenvalues are non-negative real numbers summing to one. We call such a region the *convex hull* of the eigenvalues. It will turn out that the numerical range of a normal operator is exactly this region, which we may note has the property that given any two points in the region, the line segment between them is in the region as well.

Definition 2.26. A subset M of $\mathbb{C}$ is *convex* if $\forall x, y \in M$, and for any $t \in [0,1]$, $tx + (1-t)y \in M$.

Definition 2.27. The *convex hull* of $\lambda_1, \lambda_2, \ldots, \lambda_n$, is the set of all $x \in \mathbb{C}$, such that $x = \sum_{i=1}^{n} a_i \lambda_i$, where $\sum_{i=1}^{n} a_i = 1$ and each a_i is a non-negative real number. Geometrically speaking, this will be the region enclosed by the line segments joining the λ_i's.

Theorem 2.28. *The numerical range of a normal operator $T : \mathbb{C}^n \to \mathbb{C}^n$ is the convex hull of its eigenvalues.*

Proof. Let $x \in W(T)$. Then there exists $v \in \mathbb{C}^n$ such that $\|v\| = 1$ and $x = \langle Tv, v\rangle$. Let $U = \{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of T. Then we may write $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$. Observe that

$$\begin{aligned}
1 &= \|v\|^2 \\
&= \langle v, v\rangle \\
&= \langle a_1u_1 + a_2u_2 + \cdots + a_nu_n, a_1u_1 + a_2u_2 + \cdots + a_nu_n\rangle \\
&= \langle a_1u_1, a_1u_1\rangle + \cdots + \langle a_nu_n, a_nu_n\rangle \qquad \text{(as } u_i \text{ and } u_j \text{ are orthogonal when } i \neq j) \\
&= |a_1|^2 \langle u_1, u_1\rangle + \cdots + |a_n|^2 \langle u_n, u_n\rangle \\
&= |a_1|^2 + \cdots + |a_n|^2.
\end{aligned}$$

Now observe that

$$\begin{aligned}
x &= \langle Tv, v\rangle \\
&= \langle T(a_1u_1 + a_2u_2 + \cdots + a_nu_n), a_1u_1 + a_2u_2 + \cdots + a_nu_n\rangle \\
&= \langle \lambda_1a_1u_1 + \lambda_2a_2u_2 + \cdots + \lambda_na_nu_n, a_1u_1 + a_2u_2 + \cdots + a_nu_n\rangle \\
&= \langle \lambda_1a_1u_1, a_1u_1\rangle + \cdots + \langle \lambda_na_nu_n, a_nu_n\rangle \\
&= \lambda_1|a_1|^2 \langle u_1, u_1\rangle + \cdots + \lambda_n|a_n|^2 \langle u_n, u_n\rangle \\
&= \lambda_1|a_1|^2 + \cdots + \lambda_n|a_n|^2,
\end{aligned}$$

and the theorem follows, as $|a_1|^2 + \cdots + |a_n|^2 = 1$.

□

One major theorem regarding the numerical range shows that it is always a convex set.

Theorem 2.29 (Toeplitz-Hausdorff). *Suppose T is a linear operator on the inner-product space V. Then $W(T)$ is convex.*

For a proof of the Toeplitz-Hausdorff theorem, see [?, p. 4].
Finally, we introduce the topic of *compressions*:

Definition 2.30. Let M be a subspace of V. Then $M^\perp = \{h \in V : h \perp m \; \forall \, m \in M\}$.

Theorem 2.31. *Suppose M is a subspace of V and let $h \in V$. Then there exist unique $h_M \in M$ and $h_{M^\perp} \in M^\perp$ such that $h = h_M + h_{M^\perp}$.*

Proof. Consider the set $B = \{m \in M : \; \|m\| \leq 2\|h\|\}$. B is closed and bounded, and therefore compact. Hence the continuous mapping $m \mapsto \|h - m\|$ maps B to a compact subset of $\mathbf{R}$, and so that the mapping attains its minimum on B. Hence there exists $m_0 \in B$ such that for all $m \in B$, $\|h - m\| \geq \|h - m_0\|$. Observe additionally that as $0 \in B$,

$\|h\| = \|h - 0\| \geq \|h - m_0\|$. Now suppose $v \in M \setminus B$. Then $\|v\| > 2\|h\|$. Hence,

$$\begin{aligned} \|h - v\| &= \|v - h\| \\ &\geq \|v\| - \|h\| \\ &> 2\|h\| - \|h\| \\ &= \|h\| \\ &\geq \|h - m_o\|. \end{aligned}$$

Hence, for all $v \in M$, $\|h - v\| \geq \|h - m_0\|$. Now suppose $h - m_o \notin M^\perp$. Then $\exists w \in (M)_1$ such that $\langle h - m_0, w\rangle \neq 0$. Let $u = m_0 + \langle h - m_0, w\rangle\, w$. Note that as u is a linear combination of m_0 and w, both of which are in M, $u \in M$. However,

$$\begin{aligned} \|h - u\| &= \sqrt{\langle h - m_0 - \langle h - m_0, w\rangle\, w, h - m_0 - \langle h - m_0, w\rangle\, w\rangle} \\ &\leq \sqrt{\|h - m_0\|^2 - \langle h - m_0, \langle h - m_0, w\rangle\, w\rangle - \langle\langle h - m_0, w\rangle\, w, h - m_0\rangle + \langle\langle h - m_0, w\rangle\, w, \langle h - m_0, w\rangle\, w\rangle} \\ &= \sqrt{\|h - m_0\|^2 - \overline{\langle h - m_0, w\rangle}\, \langle h - m_0, w\rangle - \langle h - m_0, w\rangle\, \langle w, h - m_0\rangle + |\, \langle h - m_0, w\rangle\, |^2\, \langle w, w\rangle} \\ &= \sqrt{\|h - m_0\|^2 - |\, \langle h - m_0, w\rangle\, |^2} \\ &< \sqrt{\|h - m_0\|^2} \\ &= \|h - m_0\|, \end{aligned}$$

a contradiction. Hence, $h - m_0 \in M^\perp$. Hence $h = h_M + h_{M^\perp}$, where $h_M = m_0 \in M$ and $h_{M^\perp} = h - m_0 \in M^\perp$. Now suppose $h = m_1 + v_1 = m_2 + v_2$, where $m_1, m_2 \in M$, $v_1, v_2 \in M^\perp$. Then

$$m_1 + v_1 = m_2 + v_2 \qquad (1)$$

$$m_1 - m_2 = v_2 - v_1 \qquad (2)$$

Hence

$$\begin{aligned} \langle m_1 - m_2, m_1 - m_2\rangle &= \langle m_1 - m_2, v_2 - v_1\rangle \\ &= 0. \qquad (\text{as } m_1 - m_2 \in M,\ v_2 - v_1 \in M^\perp) \end{aligned}$$

Hence, $m_1 - m_2 = 0$, and so $m_1 = m_2$. Equation (2) now also shows $v_1 = v_2$. Hence such a representation of h is unique.

□

Definition 2.32. Let M be a closed subspace of V. Define $P_M : V \to M$ by $P_M h = h_M$, where h_M is the unique element in M such that $h - h_M$ is in $M^\perp$. P_M is called the orthogonal projection of V onto M.

Definition 2.33. Suppose M is a closed subspace of V and that $T : V \to V$ is linear. We define the compression of T onto M, denoted T_M, to be the linear operator from M to M given by $T_M g = P_M T g$ for all $g \in M$.

Proposition 2.34. *Let M be a closed subspace of $\mathbb{C}^n$. Then P_M is self-adjoint.*

Proof. Let $v, w \in \mathbb{C}^n$. Then $v = m_1 + h_1, w = m_2 + h_2$, where $m_1, m_2 \in M$, $h_1, h_2 \in M^\perp$. Then $\langle P_M v, w\rangle = \langle m_1, m_2 + h_2\rangle = \langle m_1, m_2\rangle$. Similarly, $\langle v, P_M w\rangle = \langle m_1 + h_1, m_2\rangle = \langle m_1, m_2\rangle$. Hence, $P_M = P_M^*$.

□

Corollary 2.35. *Let T be a linear operator on V, and let M be a subspace of V. Then $W(T_M)$ is a subset of $W(T)$.*

Proof. As P_M is self-adjoint, $\forall v \in M$, $\langle T_M v, v\rangle = \langle P_M T v, v\rangle = \langle Tv, P_M v\rangle = \langle Tv, v\rangle$. Hence if $v \in (M)_1$, $\langle T_M v, v\rangle = \langle Tv, v\rangle \in W(T)$. □

We will specifically be interested in subspaces M such that $W(T_M)$ is a single point. The simplest such subspaces are the eigenspaces of T.

Example 2.36. *Suppose T is a linear operator on T and M is an eigenspace of T; that is, M is a subspace of V such that every element of M is an eigenvector of T associated with the same eigenvalue, λ. Then $W(T_M) = \{\lambda\}$.*

Proof. Let $v \in (M)_1$. Then

$$\begin{aligned} \langle T_M v, v \rangle &= \langle Tv, v \rangle \\ &= \langle \lambda v, v \rangle \\ &= \lambda \langle v, v \rangle \\ &= \lambda. \end{aligned}$$

Hence, $W(T_M) = \{\lambda\}$. □

We may also view compressions yielding single point numerical ranges as scalar multiples of the identity mapping. To show this we will require the following lemma.

Lemma 2.37. *Suppose T is a linear operator on the inner-product space V such that $\forall v \in V$, $\langle Tv, v \rangle = 0$. Then $T = 0$.*

Proof. Let $w \in V$. Then

$$\begin{aligned} 0 &= \langle T(Tw + w), Tw + w \rangle \\ &= \langle T(Tw), Tw \rangle + \langle T(Tw), w \rangle + \langle Tw, Tw \rangle + \langle Tw, w \rangle \\ &= \langle T(Tw), w \rangle + \langle Tw, Tw \rangle \end{aligned}$$

and

$$\begin{aligned} 0 &= \langle T(Tw + iw), Tw + iw \rangle \\ &= \langle T(Tw), Tw \rangle + \langle T(Tw), iw \rangle + \langle Tiw, Tw \rangle + \langle Tiw, iw \rangle \\ &= -i \langle TTw, w \rangle + i \langle Tw, Tw \rangle \\ &= i(\langle Tw, Tw \rangle - \langle TTw, w \rangle) \end{aligned}$$

so that $(\langle Tw, Tw \rangle - \langle TTw, w \rangle = 0$. Hence,

$$\begin{aligned} 0 &= \langle TTw, w \rangle + \langle Tw, Tw \rangle + \langle Tw, Tw \rangle - \langle TTw, w \rangle \\ &= 2 \langle Tw, Tw \rangle . \end{aligned}$$

Hence, $\langle Tw, Tw \rangle = 0$, and thus $Tw = 0$. Then as $w \in V$ is arbitrary, $T = 0$. □

Proposition 2.38. *Suppose T is a linear operator on V, and M is a subspace of V such that $W(T_M) = \{\alpha\}$. Then $T_M = \alpha I$.*

Proof. Let $u \in M$, $u \neq 0$. Then $v = \frac{u}{\|u\|} \in (M)_1$, so

$$\begin{aligned} \langle T_M v, v \rangle &= \alpha \\ &= \alpha \langle v, v \rangle . \end{aligned}$$

Hence

$$\begin{aligned} 0 &= \langle T_M v, v \rangle - \langle \alpha v, v \rangle \\ &= \langle T_M v - \alpha v, v \rangle \\ &= \langle (T_M - \alpha I) v, v \rangle \\ &= \frac{1}{\|u\|^2} \langle (T_M - \alpha I) u, u \rangle . \end{aligned}$$

Thus, $\langle (T_M - \alpha I) u, u \rangle = 0$ for all $u \in M$. Hence, $T_M - \alpha I = 0$ by Lemma ??, so $T_M = \alpha I$. □

It is clear that any point $\alpha \in W(T)$ is the numerical range of T_M for some 1-dimensional subspace M. We have seen that eigenspaces allow the construction of higher dimensional subspaces M such that $W(T_M)$ is a single point. Such subspaces, however, do not have to be eigenspaces, nor must the single point of the resulting numerical range be an eigenvalue of T, as the following example will demonstrate.

Example 2.39. *Let T be the linear operator on $\mathbb{C}^4$ with given by* $T = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix}$, *and let* $M = span \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. *Then $W(T_M) = \{3\}$. (Notice that T is not a normal operator, and 3 is not an eigenvalue of T.)*

Proof. Let $v \in (M)_1$. Then $v = \begin{bmatrix} 0 \\ b \\ 0 \\ d \end{bmatrix}$, where $|b|^2 + |d|^2 = 1$. Now observe that

$$\begin{aligned} \langle Tv, v \rangle &= \left\langle \begin{bmatrix} d \\ 3b \\ b+d \\ 3d \end{bmatrix}, \begin{bmatrix} 0 \\ b \\ 0 \\ d \end{bmatrix} \right\rangle \\ &= 3|b|^2 + 3|d|^2 \\ &= 3(|b|^2 + |d|^2) \\ &= 3(1) \\ &= 3. \end{aligned}$$

Hence $W(T_M) = \{3\}$.

□

The next section will begin with a discussion of when the single point numerical range of a compression of an operator must be an eigenvalue. First, however, we will conclude this section with a discussion of *corner points*, which will later be used to show that certain subspaces M with $W(T_M)$ a single point must be eigenspaces.

Definition 2.40. Let $T : V \to V$. Then $\alpha \in W(T)$ is a corner point of $W(T)$ provided that $W(T)$ is contained in the area bounded by two rays forming an angle of less than 180 degrees at their common vertex α.

Theorem 2.41. *Suppose c is a corner point of $W(T)$, and suppose $v \in (V)_1$ is such that $c =< Tv, v >$. Then $Tv = cv$; that is, c is an eigenvalue of T with associated eigenvector v.*

For a proof of the above, see [?, p. 20].

3. Results

3.1. Some Criteria for Single-Point Numerical Ranges.

If the dimension of M is large enough and $W(T_M) = \{\alpha\}$, then α must be an eigenvalue of T, as the following theorem of Fry's ([?]) demonstrates.

Theorem 3.1. *Let $T : \mathbb{C}^n \to \mathbb{C}^n$ and let M be a subspace of $\mathbb{C}^n$ such that dim $M > \frac{n}{2}$. Then if $W(T_M) = \{\alpha\}$, α is an eigenvalue of T. Moreover, the multiplicity of α must be no less than dim M – dim $M^{\perp}$.*

The second theorem from Fry's thesis ([?]) is useful in building subspaces with our desired properties.

Theorem 3.2 (Augmentation). *Let M be a proper subspace of V such that $W(T_M) = \{\alpha\}$, and let $w \in (M^{\perp})_1$ such that $\langle Tw, w \rangle = \alpha$. Let D be the set of all linear combinations of elements in M and w. Then $W(T_D) = \{\alpha\}$ if and only if $\langle Tv, w \rangle = 0 = \langle Tw, v \rangle$ for all $v \in M$.*

Definition 3.3. Given a vector v and a linear operator T, we will say w satisfies the *augmentation conditions* with v for T if $\langle Tv, w \rangle = \langle Tw, v \rangle = \langle v, w \rangle = 0$.

Notice that if w is a unit vector satisfying the augmentation conditions with v for T and $\langle Tw, w\rangle = \langle Tv, v\rangle$, the span of w and v is a subspace M such that $W(T_M)$ is a single point.

Another very similar theorem results in a corollary giving a useful criterion for determining when a compression to a subspace yields a single point. Its proof uses many of the same ideas as Fry's proof of Theorem ??.

Theorem 3.4. *Suppose M is a subspace of H, $T : H \to H$ is linear, and $U = \{u_1, u_2, \ldots, u_k\}$ is an orthonormal basis for M. Then $W(T_M)$ is a singleton if and only if $\forall i, j \in \{1, \ldots, k\}$, $i \neq j$, $\langle Tu_i, u_j\rangle = 0 = \langle Tu_j, u_i\rangle$ and $\langle Tu_i, u_i\rangle = \langle Tu_j, u_j\rangle$.*

Proof. Suppose $W(T_M) = \{\alpha\}$. Let $i, j \in \{1, \ldots, k\}$ such that $i \neq j$. Then, as $\|u_i\| = 1 = \|u_j\|$, $\langle Tu_i, u_i\rangle$, $\langle Tu_j, u_j\rangle \in W(T_M)$, and so $\langle Tu_i, u_i\rangle = \alpha = \langle Tu_j, u_j\rangle$.

Now let $v = au_i + bu_j$, where $a, b \in \mathbb{C}\backslash\{0\}$ and $|a|^2 + |b|^2 = 1$ so that $v \in (\text{span}\{u_i, u_j\})_1$. Then

$$\begin{aligned}
\alpha &= \langle Tv, v\rangle \\
&= \langle Tau_i + Tbu_j, au_i + bu_j\rangle \\
&= |a|^2 \langle Tu_i, u_i\rangle + |b|^2 \langle Tu_j, u_j\rangle + a\bar{b} \langle Tu_i, u_j\rangle + b\bar{a} \langle Tu_j, u_i\rangle \\
&= (|a|^2 + |b|^2)\alpha + a\bar{b} \langle Tu_i, u_j\rangle + b\bar{a} \langle Tu_j, u_i\rangle \\
&= \alpha + a\bar{b} \langle Tu_i, u_j\rangle + b\bar{a} \langle Tu_j, u_i\rangle
\end{aligned}$$

and hence

$$a\bar{b} \langle Tu_i, u_j\rangle + b\bar{a} \langle Tu_j, u_i\rangle = 0. \tag{3}$$

Note that the above holds for any $a, b \in \mathbb{C} \setminus \{0\}$ such that $|a|^2 + |b|^2 = 1$. Letting $a = b = \frac{1}{\sqrt{2}}$, we obtain

$$\frac{1}{2} \langle Tu_i, u_j\rangle + \frac{1}{2} \langle Tu_j, u_i\rangle = 0. \tag{4}$$

Letting $a = \frac{1}{\sqrt{2}}$, $b = \frac{i}{\sqrt{2}}$, we obtain

$$\frac{-i}{2} \langle Tu_i, u_j\rangle + \frac{i}{2} \langle Tu_j, u_i\rangle = 0, \tag{5}$$

so that

$$\frac{1}{2} \langle Tu_i, u_j\rangle - \frac{1}{2} \langle Tu_j, u_i\rangle = 0. \tag{6}$$

Now, adding equations (??) and (??), we obtain $\langle Tu_i, u_j\rangle = 0$. Subtracting these equations gives $\langle Tu_j, u_i\rangle = 0$.

Now suppose that $\forall i, j \in \{1, \ldots, k\}$, $i \neq j$, $\langle Tu_i, u_j\rangle = 0 = \langle Tu_j, u_i\rangle$ and $\langle Tu_i, u_i\rangle = \langle Tu_j, u_j\rangle = \alpha$. Let $v \in (M)_1$. Then $v = a_1u_1 + a_2u_2 + \cdots + a_ku_k$, where $|a_1|^2 + |a_2|^2 + \cdots + |a_k|^2 = 1$. Observe that

$$\begin{aligned}
\langle Tv, v\rangle &= \langle Ta_1u_1 + Ta_2u_2 + \cdots + Ta_ku_k, a_1u_1 + \cdots a_ku_k\rangle \\
&= \langle Ta_1u_1, a_1u_1\rangle + \langle Ta_2u_2, a_2u_2\rangle + \cdots + \langle Ta_ku_k, a_ku_k\rangle \\
&= |a_1|^2 \langle Tu_1, u_1\rangle + |a_2|^2 \langle Tu_2, u_2\rangle + \cdots + |a_k|^2 \langle Tu_k, u_k\rangle \\
&= (|a_1|^2 + |a_2|^2 + \cdots + |a_k|^2)\alpha \\
&= \alpha.
\end{aligned}$$

Hence, as $v \in (M)_1$ is arbitrary, $W(T_M) = \{\alpha\}$. □

Corollary 3.5. *Suppose M is a subspace of $\mathbb{C}^n$ and $T : C^n \to \mathbb{C}^n$ is linear. Then $W(T_M)$ is a singleton if and only if $\forall v, w \in M$ such that v is orthogonal to w, $\langle Tv, w\rangle = 0$.*

Proof. Suppose $W(T_M)$ is a singleton, and $v, w \in M$ are orthogonal. Then $M' = \text{span}\{\frac{1}{\|v\|}v, \frac{1}{\|w\|}w\}$ is a subspace of M, and so $W(T_{M'})$ is also a singleton. Hence, by the previous result, $\left\langle T\frac{1}{\|v\|}v, \frac{1}{\|w\|}w\right\rangle = 0$ and so $\frac{1}{\|v\|\|w\|} \langle Tv, w\rangle = 0$, and thus $\langle Tv, w\rangle = 0$.

Now suppose M is such that $\forall v, w \in M$ such that v is orthogonal to w, $\langle Tv, w\rangle = 0$. Let $U = \{u_1, u_2, \ldots, u_k\}$ be an orthonormal basis for M. Observe that for $i \neq j$, $u_i \perp u_j$, and thus by hypothesis $\langle Tu_i, u_j\rangle = 0 = \langle Tu_j, u_i\rangle$. Furthermore, let $v = u_i + u_j$ and $w = u_i - u_j$. Observe that $v, w \in M$, and that

$$\begin{aligned} \langle v, w\rangle &= \langle u_i + u_j, u_i - u_j\rangle \\ &= \langle u_i, u_i\rangle + \langle u_i, -u_j\rangle + \langle u_j, u_i\rangle + \langle u_j, -u_j\rangle \\ &= \langle u_i, u_i\rangle + \langle u_j, -u_j\rangle \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

and thus v and w are orthogonal. Hence

$$\begin{aligned} 0 &= \langle Tv, w\rangle \\ &= \langle Tu_i + Tu_j, u_i - u_j\rangle \\ &= \langle Tu_i, u_i\rangle - \langle Tu_j, u_j\rangle - \langle Tu_i, u_j\rangle + \langle Tu_j, u_i\rangle \\ &= \langle Tu_i, u_i\rangle - \langle Tu_j, u_j\rangle \end{aligned}$$

and hence $\langle Tu_i, u_i\rangle = \langle Tu_j, u_j\rangle$. Thus, by the theorem above, $W(T_M)$ is a singleton.

□

3.2. **The Sets $W_r(T)$ are Closed.** We seek to describe, for a given r, subsets of the numerical range of a linear operator such that for any point in the set, there is an r-dimensional subspace such that the numerical range of the compression of the operator to that subspace is exactly that point. Recall that we will denote such a subset by $W_r(T)$. We will ultimately conjecture that $W_r(T)$ is convex for at least normal operators; for now we will show that it must be a closed set.

Lemma 3.6. *Suppose $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=1}^{\infty}$ are sequences of vectors in $\mathbb{C}^n$ such that $(a_k)_{k=1}^{\infty}$ converges to a and $(b_k)_{k=1}^{\infty}$ converges to b. Then the sequence $(c_k)_{k=1}^{\infty}$ given by $c_k = \langle a_k, b_k\rangle$ converges to $\langle a, b\rangle$.*

Proof. Observe that

$$\begin{aligned} |c_k - \langle a, b\rangle\,| &= |\,\langle a_k, b_k\rangle - \langle a, b\rangle\,| \\ &= |\,\langle a_k, b_k\rangle - \langle a_k, b\rangle + \langle a_k, b\rangle - \langle a, b\rangle\,| \\ &= |\,\langle a_k, b_k - b\rangle + \langle a_k - a, b\rangle\,| \\ &\leq |\,\langle a_k, b_k - b\rangle\,| + |\,\langle a_k - a, b\rangle\,| \\ &\leq \|a_k\|\|b_k - b\| + \|a_k - a\|\|b\|. \end{aligned}$$

Observe that as $(a_k)_{k=1}^{\infty}$ converges to a, $\|a_k - a\|$ goes to zero as $k \to \infty$, and $\|a_k\|$ is bounded. Furthermore, $\|b\|$ is a constant and $\|b_k - b\|$ goes to zero as $k \to \infty$, so that $|c_k - \langle a, b\rangle\,|$ goes to zero as $k \to \infty$. Hence, the sequence $(c_k)_{k=1}^{\infty}$ converges to $\langle a, b\rangle$.

□

Theorem 3.7. *Suppose $T : \mathbb{C}^n \to \mathbb{C}^n$ and $W_r(T) = \{\alpha \in W(T) : \{\alpha\} = W(T_M) \text{ where } \dim(M) = r\}$. Then $W_r(T)$ is closed.*

Proof. Suppose $(\alpha_k)_{k=1}^{\infty}$ is a sequence in $W_r(T)$ converging to α. Then by Theorem ??, for each a_k there exists a set of unit vectors $H_k = \{h_{k1}, h_{k2}, \ldots, h_{kr}\}$ such that $W(T_{\mathrm{span}(H_k)}) = \{\alpha_k\}$, and if $i \neq j$,

$$\begin{aligned} \langle h_{ki}, h_{kj}\rangle &= 0 \\ \langle h_{ki}, Th_{kj}\rangle &= 0 \\ \langle h_{ki}, T^* h_{kj}\rangle &= 0. \end{aligned}$$

Now, observe that each sequence $(h_{kl})_{k=1}^{\infty}$ is bounded, as each h_{kl} is a unit vector. Hence, $(h_{k1})_{k=1}^{\infty}$ has a convergent subsequence. That is, there is some sequence $(k_j)_{j=1}^{\infty}$ such that the sequence $(h_{k_j1})_{j=1}^{\infty}$ converges. Furthermore, the sequence $(h_{k_j2})_{j=2}^{\infty}$ is bounded and so has a convergent subsequence $(h_{k_{j_s}2})_{s=2}^{\infty}$. Continuing in this manner we may find a sequence $(k_q)_{q=1}^{\infty}$ such that $(h_{k_q l})_{q=1}^{\infty}$ converges to some g_l for all $l \in \{1, 2, \ldots, r\}$. Now, observe that by our lemma, for each l, $(\langle h_{k_q l}, h_{k_q l}\rangle)_{q=1}^{\infty}$ converges to $\langle g_l, g_l\rangle$. However, as each $h_{k_q l}$ is a unit vector, $(\langle h_{k_q l}, h_{k_q l}\rangle)_{q=1}^{\infty} = (1)_{q=1}^{\infty}$ converges to 1, and thus $\langle g_l, g_l\rangle = 1$, and g_l is a unit vector. Furthermore, $(\langle Th_{k_q l}, h_{k_q l}\rangle)_{q=1}^{\infty}$ converges to $\langle Tg_l, g_l\rangle$, and hence as $(\langle Th_{k_q l}, h_{k_q l}\rangle)_{q=1}^{\infty} = (\alpha_{k_q})_{q=1}^{\infty}$ converges to α, $\langle Tg_l, g_l\rangle = \alpha$.

Furthermore, observe that for $l \neq t$, each term of the sequence $(\langle h_{k_q l}, Th_{k_q t}\rangle)_{q=1}^{\infty}$ is zero, and so as this sequence converges to $\langle g_l, Tg_t\rangle$, $\langle g_l, Tg_t\rangle = 0$, $\forall l \neq t$. Similarly, the sequences $(\langle h_{k_q l}, T^* h_{k_q t}\rangle)_{q=1}^{\infty}$ and $(\langle h_{k_q l}, h_{k_q t}\rangle)_{q=1}^{\infty}$ have all zero terms, and converge to $\langle g_l, T^* g_t\rangle$ and $\langle g_l, g_t\rangle$, respectively. Hence $\langle g_l, T^* g_t\rangle = 0$ and $\langle g_l, g_t\rangle = 0$. Therefore, by Theorem **??**, the set $\{g_1, g_2, \ldots, g_r\}$ is a basis for an r-dimensional subspace M such that $W(T_M) = \{\alpha\}$. Hence $\alpha \in W_r(T)$.

Therefore, as $(\alpha_k)_{k=1}^{\infty}$ in $W_r(T)$ is an arbitrary convergent sequence, every convergent sequence in $W_r(T)$ converges to a point in $W_r(T)$, and hence $W_r(T)$ is closed.

□

3.3. **Self-Adjoint Operators.** We will now turn our attention to the case of self-adjoint operators. Recall that the numerical range of a normal operator is the convex hull of its eigenvalues, and that the eigenvalues of a self-adjoint operator are all real numbers. Hence the numerical range of a self-adjoint operator is simply a segment of the real line.

Example 3.8. *Let* $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, *so that* T *is clearly self-adjoint and has eigenvalues* $1, 2, 3$ *and* 4. *Then* $W(T)$ *is the interval* $[1, 4]$ *of the real line.*

Proof. We could simply invoke Theorem **??**, however, we prefer to give a direct proof. Let $v \in (\mathbb{C}^4)_1$. Then $v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, where $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. Then

$$\begin{aligned} \langle Tv, v\rangle &= \left\langle \begin{bmatrix} a \\ 2b \\ 3c \\ 4d \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right\rangle \\ &= |a|^2 + 2|b|^2 + 3|c|^2 + 4|d|^2 \end{aligned}$$

and thus $\langle Tv, v\rangle$ is in the convex hull of $1, 2, 3$ and 4, which is simply the interval $[1, 4]$.

□

We observe that the example above mimics the proof of Theorem **??**, where the orthonormal basis of eigenvectors of T is simply $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. We see then that the value of $\langle Tv, v\rangle$ is simply a linear combination of the moduli of the coefficients of v represented in terms of this basis, taking as its coefficients the associated eigenvalues. This, combined with Theorem **??**, leads us to a clear method for building a subspace M whose compression yields a given point in the numerical range of T.

Example 3.9. *Let* T *be as in the previous example, and let* $M = span \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\}$. *Then* $W(T_M) = \{\frac{5}{2}\}$.

Proof. Let $v \in (M)_1$. Then $v = a \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} + b \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{2}} \\ \frac{b}{\sqrt{2}} \\ \frac{b}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} \end{bmatrix}$ where $|a|^2 + |b|^2 = 1$. Then

$$\begin{aligned} \langle T_M v, v\rangle &= \langle Tv, v\rangle \\ &= \left\langle \begin{bmatrix} \frac{a}{\sqrt{2}} \\ \frac{2b}{\sqrt{2}} \\ \frac{3b}{\sqrt{2}} \\ \frac{4a}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{a}{\sqrt{2}} \\ \frac{b}{\sqrt{2}} \\ \frac{b}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} \end{bmatrix} \right\rangle \\ &= \frac{|a|^2}{2} + \frac{2|b|^2}{2} + \frac{3|b|^2}{2} + \frac{4|a|^2}{2} \\ &= \frac{5|a|^2 + 5|b|^2}{2} \\ &= \frac{5}{2}. \end{aligned}$$

Hence, $W(T_M) = \{\frac{5}{2}\}$.

□

Notice that the basis of M in the example above consists of a linear combination $x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, such that $1 * x_1^2 + 4 * x_2^2 = \frac{5}{2}$ and a combination $y_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ such that $2 * y_1^2 + 3 * y_2^2 = \frac{5}{2}$. It is easy then to see that different choices for x_1, x_2, y_1 and y_2 will yield two dimensional subspaces whose compression gives any point in the interval $[2, 3]$. This method will not, however, work for values outside of this interval, and in fact it may be shown that there are no two dimensional subspaces yielding compressions of T with numerical range a single point outside of this interval. These ideas may be used to completely characterize the sets $W_r(T)$ for a self-adjoint operator T.

Lemma 3.10. *If $H = \{h_1, h_2, \ldots, h_k\}$ is a set of k linearly independent vectors in a space with basis $U = \{u_1, u_2, \ldots, u_n\}$, then for any set $\{i_1, i_2, \ldots, i_{k-1}\}$ of $k-1$ indices such that $1 \le i_j \le n$, span(H) contains a vector v such that v is non-zero and when v is represented in terms of the basis U as $a_1u_1 + a_2u_2 + \cdots + a_nu_n$, then $a_{i_j} = 0$ for each i_j in the set.*

Proof. Let our H and U be as described. Then any v in the span of H is given by $v = \sum_{l=1}^{k} b_l h_l$. Furthermore, each h_l may be represented as a linear combination of vectors in the basis U, so $h_l = \sum_{i=1}^{n} a_{li} u_i$, and so the coefficient of u_{i_j} in the expression of v is given by $\gamma_{i_j} = \sum_{l=1}^{k} b_l a_{li_j}$. Hence, by setting each of these sums to zero we obtain a homogeneous system of $k-1$ equations (one for each γ_{i_j}) in k unknowns (the b_l), which must have a non-trivial solution. Observe that any such non-trivial solution $(b_l)_{l=1}^{k}$ produces a vector $v = \sum_{l=1}^{k} b_l h_l$ whose coordinate vector relative to U has zeroes in positions $i_1, i_2, \ldots, i_{k-1}$. Furthermore, as there exists at least one non-zero b_l and H is linearly independent, v is non-zero. □

Lemma 3.11. *Suppose T is a normal operator on $\mathbb{C}^n$ and M is a k-dimensional subspace of $\mathbb{C}^n$. Let $L = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ be a set of eigenvalues of T, where λ_i has multiplicity m_i. Then if $\sum_{i=1}^{r} m_i \ge n - k + 1$, there exists $v \in (M)_1$ such that $\langle Tv, v\rangle$ is in the convex hull of L.*

Proof. Let $U = \{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of T. Then there is a subset $A = \{e_1, e_2, \ldots, e_s\}$ of U such that each $e_i \in A$ has as its associated eigenvalue an element in L and $s \ge n - k + 1$. Hence $|U \setminus A| \le n - (n - k + 1) = k - 1$. Now, as the dimension of M is k, by Lemma ?? there is a non-zero vector $v \in M$ such that if v is written $v = a_1u_1 + a_2u_2 + \ldots + a_nu_n$, $a_i = 0$ if $u_i \notin A$. Hence v may be written as a linear combination of elements in A, $v = b_1e_1 + b_2e_2 + \cdots + b_se_s$. We may further assume that v is a unit vector. Hence, $v \in (\text{span } A)_1$. Then

$$\begin{aligned}\langle Tv, v\rangle &= \langle T(b_1e_1 + b_2e_2 + \cdots + b_se_s), b_1e_1 + b_2e_2 + \cdots b_se_s\rangle \\ &= \langle l_1b_1e_1 + l_2b_2e_2 + \cdots + l_sb_se_s, b_1e_1 + b_2e_2 + \cdots b_se_s\rangle \\ &= l_1|b_1|^2 + l_2|b_2|^2 + \cdots + l_s|b_s|^2\end{aligned}$$

where l_i is the eigenvalue associated with e_i, and thus $l_i \in L$. Furthermore, as $\|v\| = 1$, $|b_1|^2 + |b_2|^2 + \cdots |b_s|^2 = 1$. Hence, $\langle Tv, v\rangle$ is in the convex hull of L.

□

Theorem 3.12. *Let T be a self-adjoint matrix acting on $\mathbb{C}^n$. Order the eigenvalues of T, $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k$, and let $u_1, u_2, \ldots, u_n$ be an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of T. Suppose $\alpha \in W(T)$. Let A be the set of u_i such that the eigenvalue corresponding to u_i is less than α, and let B be the set of of u_i such that the eigenvalue corresponding to u_i is greater than α. Then if M is the subspace of $\mathbb{C}^n$ of largest dimension such that $W(T_M) = \alpha$, $dim(M) = min\{|A|, |B|\} + l$, where l is the multiplicity of eigenvalue α, or 0 if α is not an eigenvalue.*

Proof. Observe that given u_{a_1} and u_{b_1} with corresponding eigenvalues λ_{a_1} and λ_{b_1}, if $\lambda_{a_1} < \alpha < \lambda_{b_1}$ (so that $u_{a_1} \in A$ and $u_{b_1} \in B$) we may construct the unit vector $w_1 = \sqrt{1 - \frac{\alpha-\lambda_{a_1}}{\lambda_{b_1}-\lambda_{a_1}}}u_{a_1} + \sqrt{\frac{\alpha-\lambda_{a_1}}{\lambda_{b_1}-\lambda_{a_1}}}u_{b_1}$. Furthermore,

$$\begin{aligned}\langle Tw, w\rangle &= \left\langle T\sqrt{1 - \frac{\alpha - \lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}}}u_{a_1} + T\sqrt{\frac{\alpha - \lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}}}u_{b_1}, \sqrt{1 - \frac{\alpha - \lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}}}u_{a_1} + \sqrt{\frac{\alpha - \lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}}}u_{b_1}\right\rangle \\ &= \left\langle \lambda_{a_1}\sqrt{1 - \frac{\alpha - \lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}}}u_{a_1} + \lambda_{a_1}\sqrt{\frac{\alpha - \lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}}}u_{b_1}, \sqrt{1 - \frac{\alpha - \lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}}}u_{a_1} + \sqrt{\frac{\alpha - \lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}}}u_{b_1}\right\rangle \\ &= \left\langle \lambda_{a_1}\sqrt{1 - \frac{\alpha - \lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}}}u_{a_1}, \sqrt{1 - \frac{\alpha - \lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}}}u_{a_1}\right\rangle + \left\langle \lambda_{b_1}\sqrt{\frac{\alpha - \lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}}}u_{b_1}, \sqrt{\frac{\alpha - \lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}}}u_{b_1}\right\rangle \\ &= \lambda_{a_1}\left|\sqrt{1 - \frac{\alpha - \lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}}}\right|^2 + \lambda_{b_1}\left|\sqrt{\frac{\alpha - \lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}}}\right|^2 \\ &= \lambda_{a_1} - \frac{\lambda_{a_1}\alpha - \lambda_{a_1}^2}{\lambda_{b_1} - \lambda_{a_1}} + \frac{\lambda_{b_1}\alpha - \lambda_{b_1}\lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}} \\ &= \lambda_{a_1} + \frac{-\lambda_{a_1}\alpha + \lambda_{a_1}^2 + \lambda_{b_1}\alpha - \lambda_{b_1}\lambda_{a_1}}{\lambda_{b_1} - \lambda_{a_1}} \\ &= \lambda_{a_1} + \frac{\alpha(\lambda_{b_1} - \lambda_{a_1}) - \lambda_{a_1}(\lambda_{b_1} - \lambda_{a_1})}{\lambda_{b_1} - \lambda_{a_1}} \\ &= \lambda_{a_1} + \alpha - \lambda_{a_1} \\ &= \alpha.\end{aligned}$$

Now, let $M_1 = \text{span}\{w_1\}$. Then any unit vector in M_1 may be written $e^{i\theta}w_1$, where $\theta \in \mathbf{R}$. As $\langle Te^{i\theta}w_1, e^{i\theta}w_1\rangle = |e^{i\theta}|^2 \langle Tw_1, w_1\rangle = 1\alpha = \alpha$, and so $W(T_{M_1}) = \alpha$. Now, if possible, construct w_2 as we constructed w_1, using u_{a_2} and u_{b_2} distinct from those used to construct w_1. Then w_1 and w_2 are orthogonal, as they are linear combinations of orthogonal vectors, and furthermore

$$\begin{aligned}\langle Tw_1, w_2\rangle &= \langle \lambda_{a_1}k_1u_{a_1} + \lambda_{b_1}k_2u_{b_1}, k_3u_{a_2} + k_4u_{b_2}\rangle \\ &= 0 \quad \text{(as distinct } u_i \text{ are orthogonal)}\end{aligned}$$

where k_i are the constants used in the construction our w_i. Similarly,

$$\begin{aligned}\langle Tw_2, w_1\rangle &= \langle \lambda_{a_2}k_3u_{a_2} + \lambda_{b_2}k_4u_{b_2}, k_1u_{a_1} + k_2u_{b_1}\rangle \\ &= 0.\end{aligned}$$

Hence by Theorem ?? , the compression of T to $\text{span}\{w_1, w_2\}$ has single point numerical range α. We may continue to construct w_i in this manner using u_{a_i} and u_{b_i} distinct from those already used to augment our space, each time preserving the single point numerical range. For all v in the space already, Tv is a linear combination of eigenvectors u orthogonal to those eigenvectors of which w_i is a linear combination, and Tw_i is a linear combination of eigenvectors u orthogonal to those used in any vector in the space, and so $\langle Tv, w_i\rangle = 0$ and $\langle Tv, w_i\rangle = 0$. We may repeat until we run out of distinct elements of A or B, and so may construct a subspace M of dimension $\min\{|\mathrm{A}|, |\mathrm{B}|\}$, where $W(T_M) = \alpha$. Furthermore, if α is an eigenvalue of T with eigenspace C of dimension l, we may augment each of the l-many u_i in C to our space, and again as all these are orthogonal to those already used, and all give $\langle Tu_i, u_i\rangle = \alpha$, the space will still have $W(T_M) = \alpha$. In this way we obtain a subspace with dimension $\min\{|\mathrm{A}|, |\mathrm{B}|\} + l$.

Now suppose the dimension of M, r, is greater than $t = min\{|\mathrm{A}|, |\mathrm{B}|\} + l$. Now let D be the larger of the sets A and B, or A if $|A| = |B|$, so that if $L = \{l_1, l_2, \dots, l_s\}$ is the set of eigenvalues associated with eigenvectors in D, and m_i is the multiplicity of l_i, then $\sum_{i=1}^{s} m_i = |D| = n - t \geq n - r + 1$. Hence by Lemma ?? there is a vector $v \in (M)_1$ such that $\langle Tv, v\rangle$ is in the convex hull of L. But then if $D = A$, $\langle Tv, v\rangle < \alpha$ and if $D = B$, $\langle Tv, v\rangle > \alpha$, a contradiction, as $v \in (M)_1$. Hence no such subspace may exist.

□

One natural question regarding the subspaces in the theorem above is whether or not they are unique. They are not; assuming, for example, that α is not an eigenvalue, there are infinitely many of them.

Proposition 3.13. *Suppose T is a self-adjoint matrix acting on $\mathbb{C}^n$, and $\alpha \in W(T)$. Then if there is a subspace M of highest possible dimension k such that $W(T_M) = \{\alpha\}$, where α is not an eigenvalue of T, there exist infinitely many such subspaces.*

Proof. Observe that by Theorem ?? there is a subspace M_1 of $\mathbb{C}^n$ with dimension k with orthonormal basis $\{a_1u_1 + a_2u_2, a_3u_3 + a_4u_4, \dots, a_{2k-1}u_{2k-1} + a_{2k}u_{2k}\}$, where each u_i is a distinct element of an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of T, such that T_{M_1} has numerical range $\{\alpha\}$. Now consider the subspace $M_2 = \operatorname{span}\{e^{i\theta}a_1u_1 + a_2u_2, a_3u_3 + a_4u_4, \cdots, a_{2k-1}u_{2k-1} + a_{2k}u_{2k}\}$, where $\theta \in (0, \pi]$. Observe that $A = \operatorname{span}\{a_3u_3 + a_4u_4, \cdots, a_{2k-1}u_{2k-1} + a_{2k}u_{2k}\}$ is a subspace of $\mathbb{C}^n$ with numerical range $\{\alpha\}$, and $h = e^{i\theta}a_1u_1 + a_2u_2$ is clearly orthogonal to each element in A, as u_1 and u_2 are orthogonal to each u_i appearing in the basis of A. Furthermore, if $w \in A$

$$\begin{aligned}
\langle Tw, h\rangle &= \langle \lambda_3 a_3 u_3 + \cdots + \lambda_{2k}a_{2k}u_{2k}, e^{i\theta}a_1u_1 + a_2u_2\rangle \\
&= 0; \\
\langle Th, w\rangle &= \langle \lambda_1 e^{i\theta}a_1u_1 + \lambda_2 a_2u_2, a_3u_3 + \cdots + a_{2k}u_{2k}\rangle \\
&= 0; \\
\|h\| &= \sqrt{|e^{i\theta}a_1u_1|^2 + |a_2u_2|^2} \\
&= \sqrt{(|e^{i\theta}||a_1||u_1|)^2 + (|a_2||u_2|)^2} \\
&= \sqrt{|a_1|^2 + |a_2|^2} \\
&= 1; \\
\langle Th, h\rangle &= \langle \lambda_1 e^{i\theta}a_1u_1 + \lambda_2a_2u_2, e^{i\theta}a_1u_1 + a_2u_2\rangle \\
&= |e^{i\theta}|^2\langle \lambda_1a_1u_1, a_1u_1\rangle + \langle \lambda_2a_2u_2, a_2u_2\rangle \\
&= \langle \lambda_1a_1u_1, a_1u_1\rangle + \langle \lambda_2a_2u_2, a_2u_2\rangle \\
&= \langle \lambda_1a_1u_1 + \lambda_2a_2u_2, a_1u_1 + a_2u_2\rangle \\
&= \langle T(a_1u_1 + a_2u_2), a_1u_2 + a_2u_2\rangle \\
&= \alpha \quad (\text{as } a_1u_1 + a_2u_2 \in (M)_1).
\end{aligned}$$

Hence by the augmentation theorem, M_2 is a subspace with numerical range $\{\alpha\}$. Now, observe that if $M_1 = M_2$, then h may be expressed as a linear combination of vectors in the basis of M_1. As our u_i are linearly independent, and h contains in its expression only u_1 and u_2, the only vector in the basis of M_1 that may have non-zero coefficient in the linear combination giving h is $a_1u_1 + a_2u_2$. However, h is clearly not a multiple of $a_1u_1 + a_2u_2$ and thus may not be so expressed. Hence, $h \notin M_1$, and thus $M_1 \neq M_2$. In this manner we may by choice of θ construct infinitely many distinct subspaces yielding compressions with numerical range $\{\alpha\}$ having dimension r. □

3.4. **Normal Operators.** It would appear that as the lemmas used in establishing the results for self-adjoint operators above are true for normal operators that methods similar to those used in Theorem ?? should extend naturally to give a similar result for general normal operators. Unfortunately, however, it quickly becomes apparent that while we may easily find the regions of the numerical range of a normal operator T in which our subsets $W_r(T)$ must be contained, establishing that a given point *must* be in $W_r(T)$ is not as simply done as in the self-adjoint case.

We will for simplicity restrict our investigation to those normal operators T on $\mathbb{C}^n$ whose numerical range is a convex $n-$gon, so that T has n distinct eigenvalues, no three of which are collinear. We will primarily be concerned with $W_r(T)$ when r is as large as possible; by Theorems ?? and ??, this means $r \leq \lfloor \frac{n}{2} \rfloor$.

To aid in our description of the n-gons that will be our numerical ranges, we introduce some new notation.

Definition 3.14. Suppose $\{a_1, a_2, \ldots, a_k\}$ is a set of complex numbers. Then we will say $a_1a_2a_3\ldots a_k$ is the set of all line segments (in $\mathbb{C}$) joining a_n to a_{n+1} for $1 \leq n < k$, together with the line segment joining a_k to a_1.

Example 3.15. *If $A = 0, B = i$, and $C = 1$, then ABC is the triangle in the complex plane with vertices* 0, *i*, *and* 1.

When n is even, we can completely describe $W_{\frac{n}{2}}(T)$.

Theorem 3.16. *Suppose T is a normal matrix acting on $\mathbb{C}^n$, $n > 2$, such that the eigenvalues of T form the convex n-gon $\lambda_1\lambda_2\ldots\lambda_n$. Then there is no subspace M of dimension greater than $\frac{n}{2}$ such that $W(T_M)$ is a single point. Furthermore, if n is even, there is a subspace M of $\mathbb{C}^n$ with dimension $\frac{n}{2}$ such that $W(T_M) = \{\alpha\}$ if and only if the set of eigenvalues of T may be partitioned into $\frac{n}{2}$ pairs such that the line segments joining the elements of each pair intersect at α.*

Proof. Suppose, in order to obtain a contradiction, that there is a subspace M of dimension larger that $\frac{n}{2}$ such that $W(T_M) = \{\alpha\}$. Then by Theorem ??, α must be an eigenvalue of T. However, as T is normal the numerical range of T must be the convex hull of its eigenvalues, and so as the numerical range of T is a convex n-gon, each eigenvalue of T is a corner point of $W(T)$, and so α is a corner point of T. Hence, by Theorem ??, any vector v such that $\langle Tv, v\rangle = \alpha$ must be an eigenvector of T, and thus M can have dimension no larger than the dimension of the eigenspace of α. However, as T acts on $\mathbb{C}^n$ and has as its numerical range an n-gon, T must have n distinct eigenvalues, one for each vertex of the n-gon, and so each eigenvalue has multiplicity 1. Thus the dimension of M is no greater that 1, a contradiction.

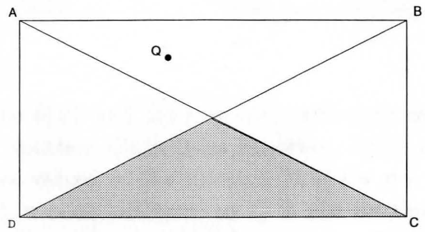
Now let $n > 2$ be even, and let $U = \{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of T. Suppose α lies on the intersection of $\frac{n}{2}$ line segments connecting distinct pairs of eigenvalues of T. Now, if α is on the line segment connecting λ_i and λ_j, there exists some t such that $\alpha = t\lambda_i + (1-t)\lambda_j$. Hence, if u_i, u_j are the eigenvectors in U with eigenvalues λ_i and λ_j, respectively, the vector $v_s = \sqrt{t}u_i + \sqrt{1-t}u_j$ clearly has norm 1 and $\langle Tv, v\rangle = t\lambda_i + (1-t)\lambda_j = \alpha$. We may find such a vector v_s for each line segment s containing α, and as each such vector will be a linear combination of eigenvectors of T orthogonal to the eigenvectors of which each other v_s is a linear combination, all such vectors will be pairwise orthogonal, and as Tv_s is simply another linear combination of the same eigenvectors as compose v_s, $\langle Tv_s, v_r\rangle = 0$ when $s \neq r$. Hence, by Theorem ??, the span of these v_s is will be a subspace M such that $W(T_M) = \{\alpha\}$, and thus as there are $\frac{n}{2}$ such v_s, this subspace has dimension $\frac{n}{2}$, as desired.

Now suppose there is a subspace M of $\mathbb{C}^n$ with dimension $\frac{n}{2}$ such that $W(T_M) = \alpha$. Then by Lemma ??, there is a unit vector $v_1 \in M$ such that $\langle Tv_1, v_1\rangle$ is in the convex hull of $\lambda_1, \lambda_2, \ldots, \lambda_{\frac{n}{2}+1}$. But there is also $v_2 \in M$ such that $\langle Tv_1, v_1\rangle$ is in the convex hull of $\lambda_1, \lambda_{\frac{n}{2}+1}, \ldots, \lambda_n$. But $\langle Tv_1, v_1\rangle = \alpha = \langle Tv_2, v_2\rangle$, and hence α must lie on the line segment joining λ_1 and $\lambda_{\frac{n}{2}+1}$. We may repeat this process for each pair of opposite eigenvalues, so that α must lie on the intersection of each of these $\frac{n}{2}$ line segments.

□

When n is odd, if $W(T_M) = \{\alpha\}$, it is clear that we may use Lemma ?? to place α in the intersection of a number of subregions of $W(T)$. However, where in the previous theorem if M has dimension $\lfloor \frac{n}{2} \rfloor$ the intersection of these regions is at most a single point, for which an appropriate subspace M exists, when n is odd the intersection of these regions contains many points for which no obvious corresponding M exists, so that it is unclear if $W_{\lfloor \frac{n}{2} \rfloor}(T)$ is the entire intersection. If $n = 3$, $\lfloor \frac{n}{2} \rfloor = 1$, and it is clear that $W_1(T) = W(T)$ (in fact, this is true for any linear operator T), and $W_r(T)$ is empty for $r > 1$. The first interesting case for n odd is that when $n = 5$. We will now provide a description of $W_2(T)$ when T is normal on $\mathbb{C}^5$, which will require a few lemmas.

Lemma 3.17. *Let $T : \mathbb{C}^4 \to \mathbb{C}^4$ be normal with eigenvalues A, B, C, D (with associated eigenvectors e_1, e_2, e_3, e_4), such that $ABCD$ is a quadrilateral in $\mathbb{C}$. Let $v \in (\mathbb{C}^4)_1$ and let $\langle Tv, v\rangle = Q$. Suppose Q is in the interior of the triangle ABC and in the interior of triangle ABD, and $w \in \mathbb{C}^4$ satisfies the augmentation conditions with v. Then $\langle Tw, w\rangle$ is in the triangles ACD and BCD.*



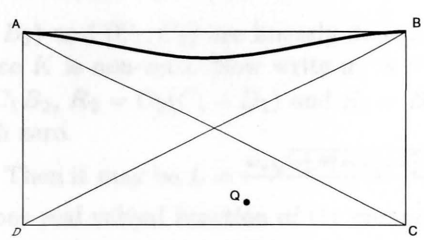
$\langle Tw, w\rangle$ MUST BE IN THE SHADED REGION.

Proof. Let $M = \text{span}\{\text{v}, \text{w}\}$, and let $\langle Tw, w\rangle = \beta$. Then if $u \in (M)_1$, $u = av + bw$ where $|a|^2 + |b|^2 = 1$. Then

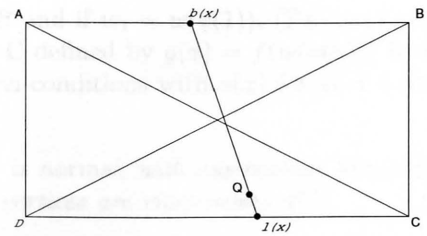
$$\begin{aligned}\langle Tu, u\rangle &= \langle Tav + Tbw, av + bw\rangle \\ &= |a|^2\langle Tv, v\rangle + a\bar{b}\langle Tv, w\rangle + b\bar{a}\langle Tw, v\rangle + |b|^2\langle Tw, w\rangle \\ &= |a|^2 Q + |b|^2\beta\end{aligned}$$

(where the final equality is due to w satisfying the augmentation conditions with v). Hence, $\langle Tu, u\rangle$ is on the line segment joining Q to β. However, by Lemma ??, as M is two dimensional it must contain a vector u_1 in the span of e_1, e_3, e_4 and u_2 in the span of e_2, e_3, e_4, so that $\langle Tu_1, u_1\rangle$ and $\langle Tu_2, u_2\rangle$ are in the triangles ACD and BCD, respectively. Hence, as these two values also must lie on the line segment connecting Q and β, β must lie in the intersection of these triangles. ∎

Lemma 3.18. *Suppose $T : \mathbb{C}^4 \to \mathbb{C}^4$ is normal, with eigenvalues A, B, C, D forming a quadrilateral $ABCD$ in $\mathbb{C}$. Suppose further that Q is in the interior of triangles DBC and ADC. Then there is a continuous path (like the one shown in the figure below) from B to A such that for each point p on the path there exists $w, v \in (\mathbb{C}^4)_1$ such that w satisfies the augmentation conditions with a unit vector v and $\langle Tv, v\rangle = Q$ while $\langle Tw, w\rangle = p$. Note that by Lemma ?? this path must lie entirely in the intersection of ABC and ABD.*



Proof. Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of eigenvectors of T corresponding to eigenvalues A, B, C and D, respectively. We may assume without loss of generality that $A = 0$. We will begin by constructing a continuous parameterization $v : [0, 1] \to (\mathbb{C}^4)_1$, such that $\langle Tv(x), v(x)\rangle = Q$, $v(0) \in \text{span}\{e_1, e_3, e_4\}$, and $v(1) \in \text{span}\{e_2, e_3, e_4\}$. Let $b(x) = A + x(B - A)$, so that b parameterizes the line from A to B. Then for $x \in [0, 1]$, the line through $b(x)$ and Q intersects the line DC at some point $l(x)$, which may be represented as a convex combination of C and D, $l(x) = D + s(x)(C - D)$. It is clear that $s(x)$ will vary continuously with $b(x)$, which is a continuous function of x.



Furthermore, the distances from Q to $b(x)$ and $b(x)$ to $l(x)$ will vary continuously with x, so that we may use the ratio of these distances to find yet another continuous function, $\alpha(x)$, such that $Q = b(x) + \alpha(x)(l(x) - b(x))$. Finally note that $s(x)$ and $\alpha(x)$ will have values in the interval $[0,1]$ for $x \in [0,1]$, by their definitions. Furthermore, $\alpha(x) \neq 0$ for $x \in [0,1]$ as then $Q = b(x)$, a contradiction, as Q is not on the segment AB. Neither can $s(x)$ be zero or one, as if it were, $l(x) = D$ or $l(x) = C$, so the segment from $l(x)$ to $b(x)$ does not pass through the interior of of triangles DBC and ADC. Now let $v(x) = (\sqrt{1-\alpha(x)})(\sqrt{1-x}e_1 + \sqrt{x}e_2) + \sqrt{\alpha(x)}(\sqrt{s(x)}e_3 + \sqrt{1-s(x)}e_4)$. Then for $x \in [0,1]$,

$$\begin{aligned}
\langle v, v\rangle &= (1-\alpha(x))(1-x)\langle e_1, e_1\rangle + (1-\alpha(x))(x)\langle e_2, e_2\rangle + \alpha(x)s(x)\langle e_3, e_3\rangle + \alpha(x)(1-s(x))\langle e_4, e_4\rangle \\
&= (1-\alpha(x))(1-x) + (1-\alpha(x))(x) + \alpha(x)s(x) + \alpha(x)(1-s(x)) \\
&= 1;
\end{aligned}$$

furthermore,

$$\begin{aligned}
\langle Tv, v\rangle &= (1-\alpha(x))(1-x)A + (1-\alpha(x))(x)B + \alpha(x)s(x)C + \alpha(x)(1-s(x))D \\
&= (1-\alpha(x))((1-x)A + xB) + (\alpha(x))(s(x)C + (1-s(x))D) \\
&= (1-\alpha(x))b(x) + (\alpha(x))l(x) \\
&= Q.
\end{aligned}$$

Now, from the definition of v, $v(0) \in \operatorname{span}\{e_1, e_3, e_4\}$, and $v(1) \in \operatorname{span}\{e_2, e_3, e_4\}$. The continuity of v follows from its construction from continuous functions.

We will now exhibit a continuous mapping $w : (v([0,1]) \to (\mathbb{C}^4)_1$ such that $w(v)$ satisfies the augmentation conditions with v, that is, $\langle Tv, w(v)\rangle = \langle T^*v, w(v)\rangle = \langle v, w(v)\rangle = 0$. Let $v = v_1e_1 + v_2e_2 + v_3e_3 + v_4e_4$ (for $v \in v([0,1])$ and let $B = B_1 + iB_2$, $C = C_1 + iC_2$ and $D = D_1 + iD_2$. Calculation shows that if $\tilde{w}(v) = w_1e_1 + w_2e_2 + w_3e_3 + w_4e_4$ where $w_1 = \frac{v_4w_4(-C_2C_1 + C_2D_1 - D_1B_2 + C_1B_2)}{v_1(B_1C_2 - C_1B_2)}$, $w_2 = \frac{v_4w_4C_2(C_1 - D_1)}{v_2(B_1C_2 - C_1B_2)}$, and $w_3 = \frac{-v_4w_4(B_1C_1 - D_1B_2)}{v_3(B_1C_2 - C_1B_2)}$, then $\tilde{w}(v)$ is a solution to the system of equations $\langle Tv, w(v)\rangle = \langle T^*v, w(v)\rangle = \langle v, w(v)\rangle = 0$. Note that $K = B_1C_2 - C_1B_2 = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}$, and so may be zero only if the vectors (B_1, B_2) and (C_1, C_2) are linearly dependent, which they are not, as $ABCD$ is a quadrilateral, and A is the origin. Hence K is non-zero. Now write $w_1 = \frac{v_4w_4R_1}{v_1K}$, $w_2 = \frac{v_4w_4R_2}{v_2K}$, and $w_3 = \frac{-v_4w_4R_3}{v_3K}$, where $R_1 = -C_2C_1 + C_2D_1 - D_1B_2 + C_1B_2$, $R_2 = C_2(C_1 - D_1)$ and $R_3 = B_1C_1 - D_1B_2$. Note that for $v \in v([0,1])$, $v_3, v_4 \neq 0$, and v_1 and v_2 are never both zero.

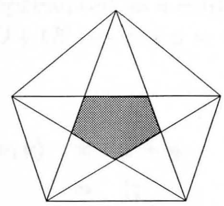
Then let $L = \sqrt{w_1^2 + w_2^2 + w_3^2 + w_4^2}$. Then it may be $L = \frac{w_4\sqrt{v_4^2R_1^2v_2^2v_3^2 + v_4^2R_2^2v_1^2v_3^2 + v_4^2R_3^2v_1^2v_2^2 + v_1^2K^2v_2^2v_3^2}}{v_1v_2v_3K}$. Note that the square root in the top is just a continuous real valued function of the components of v, call it $Y(v)$, so $L = \frac{w_4Y(v)}{v_1v_2v_3K}$. Note that as K is nonzero, $Y(v)$ is nonzero so long as v_1, v_2, and v_3 are non-zero, and that for $v \in v([0,1])$, v_3 is never 0 and v_1, v_2 are only zero at $v(1)$ and $v(0)$, respectively. Let $w(v) = \tilde{w}(v)\frac{1}{L} = \frac{v_4v_2v_3R_1}{Y(v)}e_1 + \frac{v_4v_1v_3R_2}{Y(v)}e_2 - \frac{v_4v_1v_2R_3}{Y(v)}e_3 + \frac{v_1v_2v_3K}{Y(v)}e_4$ for $v \in v((0,1))$ and let $w(v(0)) = e_2$, $w(v(1)) = e_1$, noting that these boundary values also clearly satisfy the augmentation conditions. Clearly $w(v)$ is a unit vector and by its construction must satisfy the augmentation conditions for all $v \in v([0,1])$. Furthermore w is a continuous function of v on $v((0,1))$. Now observe that as $x \to 0$, $v_2(x) \to 0$, and so $Y(v(x)) \to \sqrt{v_4^2R_2^2v_1^2v_3^2} = v_4R_2v_1v_3$ and so $w(v(x)) \to \frac{v_4v_1v_3R_2}{v_4R_2v_1v_3}e_2 = e_2 = w(v(0))$. Hence $w(v(x))$ is continuous at $x = 0$.

Similarly, as $x \to 1$, $v_1(x) \to 0$, and so $Y(v(x)) \to \sqrt{v_4^2R_1^2v_2^2v_3^2} = v_4R_1v_2v_3$ and so $w(v(x)) \to \frac{v_4v_2v_3R_1}{v_4R_1v_2v_3}e_1 = e_1 = w(v(1))$. Hence $w(v(x))$ is continuous at $x = 1$.

Now, if $w_0 = w(v(0))$, $\langle Tw_0, w_0\rangle = B$ and if $w_1 = w(v(1))$, $\langle Tw_1, w_1\rangle = A$. Now, as the function $f(u) = \langle Tu, u\rangle$ is continuous, the mapping $g : [0,1] \to \mathbb{C}$ defined by $g(x) = f(w(v(x)))$ is continuous with image joining B and A, where $w(v(x))$ satisfies the augmentation conditions with $v(x)$ for all $x \in [0,1]$.

$\square$

Theorem 3.19. *Suppose $T : \mathbb{C}^5 \to \mathbb{C}^5$ is normal, with eigenvalues forming a convex pentagon. Then $W_2(T)$ is the intersection of all quadrilaterals whose vertices are eigenvalues of T.*



If $W(T)$ is the pentagon above, $W_2(T)$ is the shaded region and its boundary.

Proof. Label the eigenvalues of T A, B, C, D and E such that the pentagon they form is $ABCDE$. Furthermore, let $\{e_1, e_2, e_3, e_4, e_5\}$ be an orthonormal basis of eigenvectors of T corresponding to eigenvalues A, B, C, D and E, respectively. Now suppose M is a two dimensional subspace of $\mathbb{C}^5$. Then M has orthonormal basis$\{u_1, u_2\}$. Thus by Lemma ?? there exists a vector $u \in (M)_1$ such that u is in the span of any 4 of e_1, e_2, e_4, e_5, and thus it is clear that $\langle Tu, u\rangle$ is in the convex hull of the 4 corresponding eigenvalues. Hence, if the numerical range of T compressed to M is a single point, that point must be in the intersection of the convex hulls of any given 4 eigenvalues, and hence $W_2(T)$ is contained in the intersection of all such quadrilaterals.

Now suppose α is in the interior of the described intersection. Then α must be in the interiors of triangles AEC, EBC, and ADB. Now consider the linear operator $T' : \mathbb{C}^4 \to \mathbb{C}^4$ with eigenvectors e_1, e_2, e_3, e_5 corresponding to eigenvalues A, B, C and E, respectively. Note that T restricted to the span of e_1, e_2, e_3, e_5 is T'. Now by Lemma ?? applied to T' there is a continuous path from A to B contained entirely in the intersection of ABC and AEB such that for any point β on the path there exists a vector w satisfying the augmentation conditions with a vector v such that $\langle T'v, v\rangle = \alpha$ and $\langle T'w, w\rangle = \beta$. As this path is continuous from A to B, and α is in the triangle ADB, the path must intersect the ray emanating from D and passing through α at some point b. Let v and w be the unit vectors satisfying the guaranteed conditions for b. Now, as α is on the line segment joining D and b, $\alpha = tD + (1-t)b$ for some $t \in [0,1]$. Let $w' = \sqrt{1-t}w + \sqrt{t}e_4$. Then

$$
\begin{aligned}
\langle Tw', w'\rangle &= \left\langle \sqrt{1-t}w + \sqrt{t}e_4, \sqrt{1-t}w + \sqrt{t}e_4 \right\rangle \\
&= (1-t)\langle Tw, w\rangle + t\langle Te_4, e_4\rangle \\
&= (1-t)\langle T'w, w\rangle + tD \\
&= (1-t)b + tD \\
&= \alpha.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\langle Tv, w'\rangle &= \langle Tv, \sqrt{1-t}w\rangle + \left\langle Tv, \sqrt{t}e_4 \right\rangle \\
&= \langle T'v, \sqrt{1-t}w\rangle + 0 \\
&= 0;
\end{aligned}
$$

$$
\begin{aligned}
\langle T^*v, w'\rangle &= \langle T^*v, \sqrt{1-t}w\rangle + \left\langle T^*v, \sqrt{t}e_4 \right\rangle \\
&= \left\langle T'^*v, \sqrt{1-t}w \right\rangle + 0 \\
&= 0;
\end{aligned}
$$

$$\begin{aligned}\langle v, w' \rangle &= \langle v, \sqrt{1-t}w \rangle + \left\langle v, \sqrt{t}e_4 \right\rangle \\ &= 0.\end{aligned}$$

Hence, the span of v and w' is a two dimensional subspace such that the compression of T to that subspace is $\{\alpha\}$.

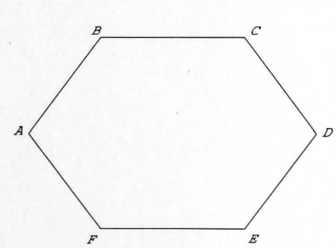
Finally, suppose α is on the boundary of the region described. Then v lies on one of the segments AC, BD, CE, or DA, and then also in the triangle EBD, ACE, BDA, or ECB, respectively. We may assume without loss of generaliy that α is on AC and in BDE. Then α may be represented as a convex combination of A and C, $\alpha = xA + yC$ or as a convex combination of B, D and E, $\alpha = rB + sD + tE$ (so, $x + y = r + s + t = 1, x, y, r, t \geq 0$). Let $v = \sqrt{x}e_1 + \sqrt{y}e_3$, and let $w = \sqrt{r}e_2 + \sqrt{s}e_4 = \sqrt{t}e_5$. Then

$$\begin{aligned}\langle v, v \rangle &= x + y \\ &= 1; \\ \langle w, w \rangle &= r + s + t \\ &= 1; \\ \langle v, w \rangle &= 0; \\ \langle Tv, w \rangle &= 0; \\ \langle T^* v, w \rangle &= \langle v, Tw \rangle \\ &= 0; \\ \langle Tv, v \rangle &= xA + yB \\ &= \alpha; \\ \langle Tw, w \rangle &= rB + sD + tE \\ &= \alpha.\end{aligned}$$

Hence, the span of v and w is a two dimensional subspace such that the compression of T to that subspace is $\{\alpha\}$. $\square$

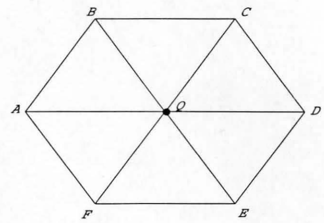
It may be noted that $W_2(T)$ described above is itself a convex pentagon, and that all other $W_r(T)$ so far found for normal operators are themselves convex sets. We hence conjecture that for $T : \mathbb{C}^n \to \mathbb{C}^n$ normal, for all r such that $W_r(T)$ is non-empty, $W_r(T)$ is convex.

We will conclude with an informal investigation of $W_r(T)$ for normal operators on $\mathbb{C}^6$. Suppose $T : \mathbb{C}^6 \to \mathbb{C}^6$ is normal and has eigenvalues A, B, C, D, E, and F forming a convex hexagon $ABCDEF$. Then $W_1(T) = W(T)$ is the hexagon shown below.



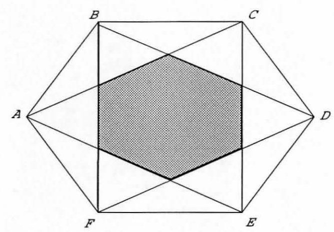
$W(T)$

Now by Theorem ??, $W_3(T)$ is nonempty if and only if the line segments AD, BE and CF intersect at some point Q as shown below, in which case $W_3(T) = \{Q\}$.



$W_3(T) = \{Q\}$

Finally, using Lemma ?? as in the proofs of Theorems ?? and ??, it is clear that $W_2(T)$ must be contained in the intersection of all possible pentagons formed by any five of A, B, C, D, E and F, the shaded region in the figure below.



To see that $W_2(T)$ is exactly the shaded region above, notice that any point α in the region is in the intersection of the triangles ACE and BDF. Hence, $\alpha = x_1A + x_2C + x_3E = y_1B + y_2D + y_3F$, where x_1, x_2, x_3, y_1, y_2 and y_3 are non-negative real numbers such that $x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 1$. Now, as T is normal, there is an orthonormal basis for $\mathbb{C}^6$ consisting of eigenvectors of T, $\{u_A, u_B, u_C, u_D, u_E, u_F\}$, where u_i is has as its corresponding eigenvalue i. Then the vectors v and w given by $v = \sqrt{x_1}u_A + \sqrt{x_2}u_C + \sqrt{x_3}u_E$ and $w = \sqrt{y_1}u_B + \sqrt{y_2}u_D + \sqrt{y_3}u_F$ are orthogonal unit vectors such that $\langle Tv, w\rangle = \langle Tw, v\rangle = 0$ and $\langle Tw, w\rangle = \langle Tv, v\rangle = \alpha$. Hence if $M = \text{span}\{\text{v}, \text{w}\}$, M has dimension 2 and $W(T_M) = \{\alpha\}$. Thus we may completely describe $W_r(T)$ for all r such that $W_r(T)$ may be non-empty, when T is as described above. Notice that each $W_r(T)$ described above is convex, further supporting our conjecture.

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