JULIA SETS: AN INTRODUCTION TO CHAOS IN THE PLANE

Honors Thesis in Mathematics

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This work is dedicated to future generations of Washington and Lee mathematicians.

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INTRODUCTION

Within the past five years the scientific community has enthusiastically begun to employ a new tool which has been popularly referred to as chaos theory. This tool helps scientists to identify order, at least qualitatively, in dynamical systems which appear random, or chaotic. Examples of chaotic dynamics include the turbulent flow of fluids, weather, phase transitions, and population fluctuations. Quantitative analysis of such phenomena has evaded scientists; consequently, they have turned to qualitative analysis. In addition to providing insight into the dynamics of chaotic systems, the new tool has caused many scientists to question whether or not quantitative analysis is possible.

Many individuals from an interestingly diverse collection of fields have independently developed pieces of chaos theory. Some authors have attempted to present the history of this development and to explain simply the implications of the new theory. In his book Chaos: Making a New Science, James Gleick presents the subject interestingly to the non-scientific community. Gleick's text inspired the author to study in greater depth one particular aspect of chaos theory, namely Julia sets.

In this paper we introduce the reader to a family of functions which give rise to Julia sets. Making their home in the complex plane, these sets exhibit many interesting properties, including chaotic dynamics. The aesthetic quality of Julia sets help us to appreciate more fully the properties which these sets exhibit. We encourage the reader to flip through the graphs presented in Appendix B, evaluating their artistic value now and their scientific value later.

The study of these sets began during World War I. The French mathematicians Gaston Julia and Pierre Fatou wrote extensive papers, Julia in 1918 and Fatou in 1919 and 1920, which established the fundamental definitions and theorems of this paper. Largely due to the increased computational power and graphical ability of modern computers, the subject has recently experienced a flood of attention. Among the current mathematicians studying Julia sets, Sullivan has classified the dynamics in the stable set (defined in this paper) while Douady, Hubbard, and Mandelbrot have analyzed the dynamics of quadratic polynomials.

This work merely introduces the reader to the subject of Julia sets, which has been explained in much greater depth by more capable mathematicians. In addition to omitting much of the established work of the mathematicians mentioned above, we exclude a great deal of current work on the subject. The references should provide the interested reader with some starting points to pursue the subject in greater detail.

ACKNOWLEDGEMENTS

I thank the professors and students who taught me mathematics, especially Dr. Wayne Dymacek. The following work would have been impossible without his continuous support.

1. REVIEW OF COMPLEX NUMBERS

For those readers who have forgotten everything they learned about complex numbers, whether intentionally or otherwise, we begin with a brief review. (The less forgetful reader should move on to the next section.) A **complex number** z has the form z = a + bi, where a and b are real numbers and i is defined to be $\sqrt{-1}$. Note that $i^2 = -1$. The **real part** of z is a, and b is called the **imaginary part** of z. The set of complex numbers **C** consists of every possible combination of real and imaginary parts.

To help visualize these numbers, we may represent complex numbers as points (a,b) in the popular Cartesian plane, with the real part on the horizontal axis and the imaginary part on the vertical axis. Since we are representing complex numbers, we will call this space the complex plane. Consult Figure 1.

As a bonus, we may determine points by ordered pairs not only of horizontal and vertical distance but also of radius and angle. Thus, the ordered pair (r,Θ) represents a point in the complex plane, where the radius r measures the distance between the point and the origin (0,0), and the angle Θ measures in a counter-clockwise direction the angle in radians between the line connecting the point with the origin (0,0) and the real axis.

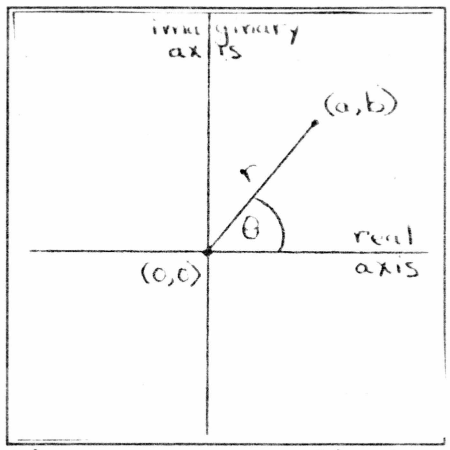

Figure 1: Coordinates of the complex number z = a + bi in the complex plane

Using our rusty trigonometry skills, we find formulas which relate a complex number in **rectangular form** (a,b) to the same number in **polar form** (r,Θ):

$$a = r\cos\Theta,$$

$$b = r\sin\Theta,$$

$$r = \surd(a^2 + b^2), \text{ and}$$

$$\Theta = \arctan(a/b).$$

Thus, a complex number z may have the form $z = a + bi$ or the form $z = r(\cos\Theta + i\sin\Theta)$.

As with real numbers, complex numbers may be added, subtracted, multiplied, and divided in pairs to obtain other complex numbers. The rules for addition and multiplication follow. Given any two complex numbers $x = a + bi$ and $y = c + di$, where a, b, c, and d are real numbers,

$$x + y = (a + bi) + (c + di) = (a + c) + (b + d)i \text{ and}$$

$$xy = (a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Subtraction and division are simply the inverse operations of addition and multiplication, respectively. For every non-zero complex number z there exists a unique additive inverse -z and a unique multiplicative inverse 1/z. Thus, subtraction and division can be defined in terms of addition and multiplication as follows:

$$x - y = x + (-y) \text{ and}$$

$$x/y = x(1/y).$$

In addition to these binary operations, one may also find the absolute value or the square root of a complex number $z = a + bi$ as follows:

$$|z| = \sqrt{(a^2 + b^2)} \text{ and}$$

$$\sqrt{z} = \sqrt{[(a + \sqrt{(a^2 + b^2)})/2]} + (\text{sgn } b) i \sqrt{[(-a + \sqrt{(a^2 + b^2)})/2]}$$

Note that the absolute value equals the radius of the complex number in polar form. Also, the square root is defined to be positive, and the value (sgn b) is defined to be +1 if $b > 0$ and -1 if $b < 0$.

2. FAMILIES OF FUNCTIONS

The setting for the sets which we will study is the **Riemann sphere**. This surface is simply the union of the complex plane with the point at infinity: $\mathbf{C}^* = \mathbf{C} \cup \{\infty\}$. As the name implies, one may picture this surface as a sphere. Zero corresponds to the south pole; infinity to the north pole. Lines of the form $y = |z_0|$, where z_0 is a fixed point in $\mathbf{C}^*$, correspond to latitude lines whose distance from the south pole is z_0.

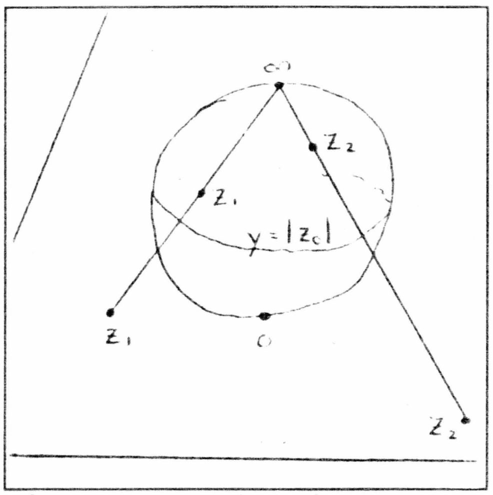

Figure 2 Projection of the Riemann sphere onto the complex plane

As visual images tend to enlighten the mind as well as please the eye, we present how the Riemann sphere can be constructed from the complex plane. Imagine a sphere mounted precariously upon a plane which extends as far as the eye can see. Consult Figure 2. The origin marks the point of contact between the two surfaces. Pick any point on the plane and draw a line through that point and the north pole of the sphere. The line will intersect the sphere in exactly one point so that a one-to-one correspondence exists between every point on the plane and every point except the north pole on the sphere. As mentioned, the north pole represents the infinity point.

When studying Julia sets in general, mathematicians ususally examine the family of rational functions in the home of the Riemann sphere.

Definition 2.1: A **rational function** R: $\mathbf{C}^* \to \mathbf{C}^*$ has the form R(z) = p(z)/q(z), where p(z) and q(z) are polynomials with complex coefficients and no common factors. The **degree** of R measures how many times R wraps the Riemann sphere around itself, and this number is simply the maximum of the degrees of p and q.

The work of Julia and Fatou applies to rational functions whose degree is at least two. Since we wish to examine some of the results of Julia and Fatou while keeping the study as simple as possible, we limit ourselves to the familiar family of functions called the quadratic polynomials. Note, however, that many of the theorems in this paper apply to rational functions in general.

Definition 2.2: A **quadratic polynomial** is a function p: $\mathbf{C}^* \to \mathbf{C}^*$ of the form $p(z) = az^2 + bz + c$, where a, b, c $\in \mathbf{C}$ and $a \neq 0$.

Ever striving for simplicity, we present the following theorem which allows us to study the properties of the whole family of quadratic polynomials while only examining a certain subfamily of these functions.

Theorem 2.3: A function $p: \mathbf{C}^* \to \mathbf{C}^*$ of the form $p(z) = az^2 + bz + c$, where $a, b, c \in \mathbf{C}$ and $a \neq 0$, can be transformed by the function $z' = q(z) = (1/\surd a)z - b/(2a)$ into a function of the form $p(z') = (z')^2 + c'$, for some $c' \in \mathbf{C}$.

Proof: $p(z') = p[q(z)] = p[(1/\surd a)z - b/(2a)]$

$$= a[(1/\surd a)z - b/(2a)]^2 + b[(1/\surd a)z - b/(2a)] + c$$

$$= a[(1/a)z^2 - (b/a^{3/2})z + b^2/(4a^2)] +$$

$$(b/\surd a)z - b^2/(2a) + c$$

$$= z^2 - (b/\surd a)z + b^2/(4a) + (b/\surd a)z - b^2/(2a) + c$$

$$= z^2 - b^2/(4a) + c. \qquad \text{Q.E.D.}$$

This proof establishes that $c' = - b^2/(4a) + c$. Observe that q is simply a linear function so that the composition of p with q rotates, shifts, and magnifies or shrinks the image of p. Now rotating, shifting, and magnifying or shrinking--as opposed to say bending, ripping, and tearing--do not alter the properties of the images of p which we will study. As a result, we delightedly limit our family of quadratic polynomials to $\{p: \mathbf{C}^* \to \mathbf{C}^* \mid p(z) = z^2 + c \;\; \forall\, c \in \mathbf{C}\}$.

3. AN APPROACH TO JULIA SETS

Iteration begins this approach to Julia sets. The process is simple enough. Choose a complex value z_0 and a quadratic function $p(z) = z^2 + c$, for some particular complex number c. Find a number-crunching computer and calculate $p(z_0)$, $p(p(z_0))$, $p(p(p(z_0)))$, etc. For convenience, we write $p^n(z) = p(p(\ldots p(z)\ldots))$, where the iteration on the right is repeated n times. Be careful not to confuse this notation with similar notation for products and derivatives: $p^2(z) \neq p(z)p(z)$ and $p^2(z) \neq p''(z)$!

In each of the definitions which follow, we will consider a point $z_0 \in \mathbf{C}^*$ and a quadratic polynomial $p\colon \mathbf{C}^* \to \mathbf{C}^*$ given by $p(z) = z^2 + c$, for some $c \in \mathbf{C}$.

Definition 3.1: The **forward orbit** of z_0 is defined to be the sequence of iterates $O^+(z_0) = (z_0, p(z_0), p(p(z_0)), \ldots, p^n(z_0), \ldots)$. The **inverse orbit** of z_0 is defined to be the sequence of positive iterates of the inverse function p^{-1}, $O^-(z_0) = (z_0, p^{-1}(z_0), p^{-1}(p^{-1}(z_0)), \ldots, p^{-n}(z_0), \ldots)$.

Note that p^{-n} represents the n-th iteration of the inverse function and that in constructing the inverse orbit of a point, we take the inverse function as the positive square root: $p^{-1}(z) = +\sqrt{(z - c)}$.

After writing out a few iterates of a point, one wonders where the sequence is headed. Will the sequence tend to a fixed number, will it tend toward infinity, will it tend toward a fixed cycle of numbers, or will it simply wander aimlessly?

In anticipation of the answers to these questions we present the following definitions.

Definition 3.2: If $p^n(z_0) = z_0$ for some natural number n and some $z_0 \in \mathbf{C}$, then z_0 is called a **periodic point.** If $n = \inf\{i \in \mathbf{N} \mid p^i(z_0) = z_0\}$, then n is called the **period** of $O^+(z_0)$. If $n = 1$ then z_0 is called a **fixed point.** Further, $O^+(z_0)$ is said to be **eventually periodic** if $p^n(z_0)$ is a periodic point for some n.

Periodic points can be classified according to their eigenvalues.

Definition 3.3: The **eigenvalue** $L(z_0)$ of a periodic point $z_0 \in \mathbf{C}$ with period n is the derivative of the n-th iterate of p, i.e., $L(z_0) = (p^n)'(z_0)$. The orbit of z_0 is then said to be

1) **attracting** if $0 \leq |L| < 1$,

2) **repelling** if $|L| > 1$, or

3) **neutral** if $|L| = 1$.

These terms describe how other points react to the orbit of a periodic point, whether the periodic orbit attracts, repels, or remains indifferent to other points. Neutral points are so difficult as to merit being ignored in this paper.

To illustrate some of these definitions, consider the simplest polynomial in the family, namely $p(z) = z^2$. We can find an abundance of nonperiodic orbits; for example,

$$O^+(2) = (2,4,16,256,\ldots),$$

$$O^+(1/2) = (1/2,1/4,1/16,\ldots), \text{ and}$$

$$O^+(1 + i) = (1 + i,2i,-4,16,\ldots).$$

Of periodic points, p has three fixed: 0, 1, and ∞. In order to find points of period two or higher, recall from the depths of complex analysis the n n-th roots of unity: the solutions to the equation $w^n = 1$, are 1, w_n, w_n^2, ..., w_n^{n-1}, where

$$w_n = \cos(2\pi/n) + i\sin(2\pi/n).$$

For example, the three cube roots of unity are 1, $w_3 = (-1 + i\sqrt{3})/2$, and $w_3^2 = (-1 - i\sqrt{3})/2$. Thus, an example of a periodic orbit of period two is $O^+(w_3) = (w_3,w_3^2,w_3^4 = w_3,\ldots)$. The orbit of the first fifth root of unity demonstrates a period of four: $O^+(w_5) = (w_5, w_5^2, w_5^4, w_5^8 = w_5^3, w_5^6 = w_5, \ldots)$.

Knowing which points are periodic, we can calculate their eigenvalues:

$$L(0) = p'(0) = 0,$$

$$L(1) = p'(1) = 2,$$

$$L(w_3) = (p^2)'(w_3) = 4, \text{ and}$$

$$L(w_5) = (p^4)'(w_5) = 16.$$

Thus, 0 is an attracting point, and 1, w_3, and w_5 are repelling points. We also say that ∞ acts as an attracting point.

In order to define the Julia set of a polynomial, we need to review the definition of closure. For completeness, we define some closely related terms as well.

Definition 3.4: The **closure** of a set A is the set of all points a such that A contains points arbitrarily near to a. A set is **closed** if it is identical with its closure. A set is **open** if its complement is closed. The **exterior** of a set is the complement of its closure. The **interior** of a set is the exterior of its complement. The **boundary** of a set is the intersection of its closure with the closure of its complement.

We finally give a definition of Julia set.

Definition 3.5: The **Julia set** of p, denoted J(p), is the closure of the set of repelling periodic points of p. The **stable set** of p, denoted S(p), is the complement of the Julia set, i.e., $S(p) = \mathbf{C}^* - J(p)$.

In our example using $p(z) = z^2$, the Julia set includes 1 and the various roots of unity. As we will see later, the Julia set is exactly the unit circle. Consequently, the entire Riemann sphere, except the unit circle, constitutes the stable set.

The stable set gets its name from the fact that it contains all of the attracting points, which may be fixed or periodic points, and that every point in the stable set tends toward one of these attracting points or orbits. We conveniently group points in the stable set which have the same attractor.

Definition 3.6: If z_0 is an attractive, fixed point, then its **basin of attraction** is defined to be the set $A(z_0) = \{z \in \mathbf{C}^* \mid p^k(z) \to z_0 \text{ as } k \to \infty\}$. If P is an attractive orbit of period n, say $P = (z_0, z_1, \ldots, z_{n-1}, z_0, \ldots)$, then each periodic point z_i, $i = 0,1,\ldots,n-1$, has a basin of attraction, and the basin of attraction of the orbit is the union of the basins of attraction of each point, i.e., $A(P) = A(z_0) \cup A(z_1) \cup \cdots \cup A(z_{n-1})$.

Returning to our example, all of the points inside the unit circle lie in the basin of attraction of 0 while all points outside the unit circle lie in the basin of attraction of ∞. The Julia set, which is the unit circle, marks the boundary between these two basins of attraction.

4. ANOTHER APPROACH TO JULIA SETS

This alternate approach to Julia sets is actually the approach taken by Julia and Fatou. Complex analysts take great delight in the definitions and theorems which are presented here, and indeed they should, for such definitions allow us to prove many of the theorems which characterize Julia sets and stable sets. As usual, we begin with a review of some definitions.

Definition 4.1: Consider a family of rational functions $F = \{R_i: \mathbf{C}^* \to \mathbf{C}^* \mid i \in I\}$, for some index set I. The sequence of values $\{R_n(z)\}$ is said to **converge uniformly** to R(z) if $\forall\ \epsilon > 0$ there exists an integer N such that when $n > N$, $|R(z) - R_n(z)| < \epsilon$ $\forall\ z \in \mathbf{C}^*$.

For those readers less familiar with this concept, note that the definition of uniform convergence closely resembles the definition of convergence. The only difference is that the N in the definition of uniform convergence exists independently of the value of z, whereas the N in the definition of convergence depends upon the value of z. Note that every sequence which converges uniformly also converges, although the converse is not true. Thus, uniform convergence specializes the concept of convergence.

Without defining compactness, recall that a set is **compact** if and only if it is closed and bounded.

We now introduce a definition which characterizes a family of functions in a way unfamiliar to most of us.

Definition 4.2: Let G be an open subset of $\mathbf{C}^*$. The family $\{R_i: G \to \mathbf{C}^* \mid i \in I\}$ is said to be a **normal family** if every sequence

$\{R_n\}$ contains a subsequence $\{R_{n_k}\}$ which converges uniformly on compact subsets of G.

Before using this new term to define the stable set and the Julia set, let us try to relate this characteristic of a family of functions to another characteristic which is more familiar.

Definition 4.3: The family of rational functions $\{R_i: \mathbf{C}^* \to \mathbf{C}^* \mid i \in I\}$ is an **equicontinuous family** if given $\epsilon > 0$, there exists $\delta > 0$ such that $|z_1 - z_2| < \delta$ implies that $|R_i(z_1) - R_i(z_2)| < \epsilon \ \forall i \in I$.

Note that each function in an equicontinuous family is itself continuous, which is a more familiar property than normality. The following theorem allows us to equate normal families with equicontinous families.

Arzela's Theorem 4.4: Let G be an open subset of $\mathbf{C}^*$. The family $\{R_i: G \to \mathbf{C}^* \mid i \in I\}$ is a normal family if and only if it is an equicontinuous family on every compact subset of G.

We now present an alternate definition of the Julia set.

Definition 4.5: Given a polynomial p: $\mathbf{C}^* \to \mathbf{C}^*$ such that $p(z) = z^2 + c$ for some $c \in \mathbf{C}$, a point $z \in \mathbf{C}^*$ is said to be an element of the **stable set** S(p) if there exists a neighborhood G of z such that the family of iterates $\{p^n|G\}$, restricted to G, is a normal family. The **Julia set** J(p) is the complement of the stable set. Points in the Julia set, then, have no neighborhood in which $\{p^n|G\}$ is a normal family.

For the example $p(z) = z^2$, we can show that points lying outside the unit circle lie within S(p).

Theorem 4.6: $\{z \in \mathbf{C}^* \mid |z| > 1\} \subset S(p)$, where $p(z) = z^2$.

Proof: Let z be any point in $\mathbf{C}^*$ such that $|z| > 1$ and let U_z be a small, open disk of radius $\delta > 0$ about z. Consider the family of iterates $\{p^n(z)|U_z\}$. We wish to show that the sequence $\{p^n(z)\} = \{z^{2^n}\}$ converges uniformly to $f(z) = \infty$ on compact subsets of U_z.

Let $\epsilon > 0$ be given, let K be any compact subset of U_z, and let z_0 be the point in K farthest from $z = \infty$ and, hence, closest to $z = 0$. Thus, for every z in K, $|z_0| \le |z|$. Since $|z| > 1 \;\forall\; z \in K$, $|z_0|^{2^n} \le |z|^{2^n}$ so that $|z^{2^n} - \infty| \le |z_0^{2^n} - \infty|$. Now for n large, $|z_0^{2^n} - \infty| < \epsilon$ so that $|z^{2^n} - \infty| < \epsilon \;\forall\; z \in K$. Q.E.D.

Furthermore, we can show that the Julia set of $p(z) = z^2$ is exactly the unit circle.

Theorem 4.7: $\{z \in \mathbf{C}^* \mid |z| = 1\} = J(p)$, where $p(z) = z^2$.

Proof: We need to show that the family of iterates $\{p^n\}$ is not equicontinous on any open set which intersects the circle. That is, given $\delta > 0$ we need to show that, $\forall\; \epsilon > 0$ and any two points $z_1, z_2 \in J(p)$, $|z_1 - z_2| < \delta$ implies that $|p^n(z_1) - p^n(z_2)| > \epsilon$ for some n. Let $\delta > 0$ be given and let $z_1 = e^{i\Theta_1}$ and $z_2 = e^{i\Theta_2}$, where Θ_1 and Θ_2 are angles in radians. Note that the angle between Θ_1 and Θ_2 is approximately equal in size to δ. Now applying the function p to z_1 and z_2 simply doubles the size of the angles Θ_1 and Θ_2: $p(e^{i\Theta_1}) = e^{i2\Theta_1}$ and $p(e^{i\Theta_2}) = e^{i2\Theta_2}$. Hence, the angular difference between the two points also doubles so that $\forall\; \epsilon > 0$ we can find an n large enough so that $|p^n(z_1) - p^n(z_2)| = |e^{i2^n\Theta_1} - e^{i2^n\Theta_2}| > \epsilon$. Q.E.D.

5. PROPERTIES OF JULIA SETS

For the remainder of this paper, we adopt the first definition of the Julia set. The primary reason for introducing the second definition is that this approach, used by Julia and Fatou, is used more frequently in proofs of the theorems which describe the properties of the Julia set and the stable set. As we will see, however, the two definitions are equivalent. But whether we consider the Julia set as the closure of the set of repelling periodic points or as the set of points on which the family of iterates $\{p^n\}$ is not normal, the properties of the Julia set are the same. These properties have become the principal interest for those who study Julia sets and stable sets.

Perhaps the most important property of any set is whether or not it contains anything.

Theorem 5.1: $J(p)$ is nonempty.

Proof: Suppose $J(p) = \emptyset$. Then $S(p) = \mathbf{C}^*$, and therefore, the family $\{p^n\}$ is normal in $\mathbf{C}^*$. Hence, there exists a subsequence $\{p^{n_k}\}$ which converges uniformly to a limit function P. Now, P is a continuous function on the Riemann sphere so that deg(P) is finite. But, $\deg(p^{n_k}) \to \infty$ as $n_k \to \infty$ and $\deg(p^{n_k}) \to \deg(P)$ as $n_k \to \infty$, a contradiction. Q.E.D.

As promised, we present a theorem which shows that the two definitions of Julia set are equivalent. Here, the Julia set is defined as the closure of the set of repelling periodic points.

Theorem 5.2: $J(p) = \{z \mid \{p^n\} \text{ is not normal at } z\}$.

The rather involved proof of this theorem is found in Robert

Devaney's An Introduction to Chaotic Dynamical Systems. The proof provides us with two additional properties of the Julia set.

Corollary 5.3: J(p) is a perfect set; i.e., every point in J(p) is a limit point of other points in J(p).

Corollary 5.4: J(p) is completely invariant; i.e., if $z \in J(p)$, then $p(z)$, $p^{-1}(z) \in J(p)$.

As Devaney remarks, a modification of the proof of the above theorem leads to another, useful theorem and its corollary.

Theorem 5.5: Let $z_0 \in J(p)$. Then $J(p) = \text{closure}(\cup_{k=0}^{\infty} p^{-k}(z_0))$.

Corollary 5.6: J(p) has empty interior.

This theorem provides us with an algorithm for computing Julia sets graphically. Suppose we want to generate a graph of the Julia set of the polynomial $p(z) = z^2 + c$, where c is some complex number. First, we find a point z_0 in the Julia set using the definitions of periodic and repelling, and plot this point. Next, using the definition of the square root of a complex number, we calculate the inverse image of z_0, which gives us two points: $p^{-1}(z_0) = \pm\sqrt{(z_0 - c)}$. Plot these two points. Repeat the procedure for each new pair of points.

In actually programming a computer to run this algorithm, one confronts the problem of keeping track of all the new points generated. With each step of the algorithm the number of inverse images doubles so that the program would soon eat up gobs of memory. In order to solve this problem, we used a "random" function to select either the positive or the negative inverse image during each repetition of the algorithm. Consult Appendix

A for a Turbo Basic program, inspired by Michael Barnsley, which employs this random algorithm.

Note that the instructions to the program listed in the appendix assert that the initial value need not be a point in the Julia set. We have found this assertion to be correct on many trials. The program generates the same graph regardless of whether the initial value is in the Julia set or the stable set, provided the first ten points are not plotted. We again encourage the reader to examine some of the graphs provided in Appendix B. These graphs were generated on an IBM pc using the program given in Appendix A.

The Julia set exhibits the following property as well.

Theorem 5.7: $J(p)$ is locally eventually onto; i.e., if G is an open subset of $\mathbf{C}^*$ such that $G \cap J(p) \neq \emptyset$, then there exists an N such that $J(p) = p^N(G \cap J(p))$.

This theorem states that if we select an arbitrarily small, open piece of a Julia set, then we can generate the entire Julia set simply by iterating the polynomial p enough times on the selected domain. Notice how this property resembles the phenomenon of cell division. A human being begins as one cell and after many repetitive divisions becomes a fetus. Granted, growth involves much more complicated actions, such as specialization of cells, but it is interesting to note the similarities between mathematics and biology.

The final property we present to encourage the reader to pursue the study of chaotic dynamics, which has become quite popular in modern mathematical and scientific research.

Theorem 5.8: p is chaotic on J(p); i.e., p has the following properties:

1) sensitive dependence on initial conditions,

2) topological transitivity, and

3) dense periodic points.

We comment on the first and third properties of chaos. That p depends sensitively on initial conditions means qualitatively that if we choose two points in J(p) which are close together, then the orbits of these two points will tend to spread arbitrarily far apart. That p has dense periodic points means that between any two points in J(p) lies a third point in the set.

Notice how the property of sensitive dependence on initial conditions might affect the prediction ability of scientific models. Suppose a meteorologist develops a chaotic dynamical model for the weather consisting of a number of differential equations. Suppose further that he observes some initial conditions, say temperature and pressure, using instruments which inherently have a small degree of error in them. If the meteorologist attempts to predict future weather by iterating the equations using these measured values for the initial conditions, then the further into the future he trys to predict, the more inaccurate his predictions will be. This phenomenon is referred to as the "butterfly effect" because, presumably, the flapping of a butterfly's wings, representing the error in initial conditions, could lead to an unpredicted hurricane next month. If weather systems really behave chaotically, as such a model predicts, then meteorologists will never have the ability to forecast weather far into the future because there will always be a small degree of error in measuring the initial conditions.

6. CLASSIFICATIONS OF JULIA SETS

Having described some of the properties common to the Julia sets of all polynomials in the family $\{p:\mathbf{C}^* \to \mathbf{C}^* \mid p(z) = z^2 + c$, for some $c \in \mathbf{C}\}$, we now present two means of categorizing these Julia sets. First, the structure of the stable set determines the type of Julia set.

Theorem 6.1: The stable set of p consists of one, two, or infinitely many connected components.

The graphs in Appendix B demonstrate each of these three possibilities. The "cauliflower" Julia set of $p(z) = z^2 + 0.4$ shows that the stable set consists of one connected component engulfing the broken pieces of the Julia set. The Julia set of $p(z) = z^2 - 0.5$ divides the stable set into two distinct components. The Julia set of $p(z) = z^2 - 1.25$ separates the stable set into infinitely many connected components.

The second method of categorizing Julia sets comes about through an examination of the orbit of the critical point $z_0 = 0$ as follows:

1) if $O^+(0) \to \infty$, then J(p) is a Cantor set,

2) if $O^+(0) \to$ attracting fixed or periodic point, then J(p) is the closure of one, two, or infinitely many Jordan curves, and

3) if $O^+(0)$ is eventually periodic, but not periodic, then J(p) is a dendrite.

Of course, the categories remain meaningless until we define them.

Definition 6.2: A **Cantor set** is any collection of points in C^* which is closed, totally disconnected, and perfect.

A set is totally disconnected if each connected component consists of only one point, and as stated previously, a perfect set contains all of its limit points. To construct the classic example of a Cantor set, begin with the closed interval [0,1] on the real line, remove the middle third of this interval so that the set [0,1/3] U [2/3,1] remains, remove the middle thirds of each of these intervals, and repeat this process <u>ad infinitum</u>. Consult Figure 3. The resulting set, although not much to look at, does contain some points, 0 and 1 for instance.

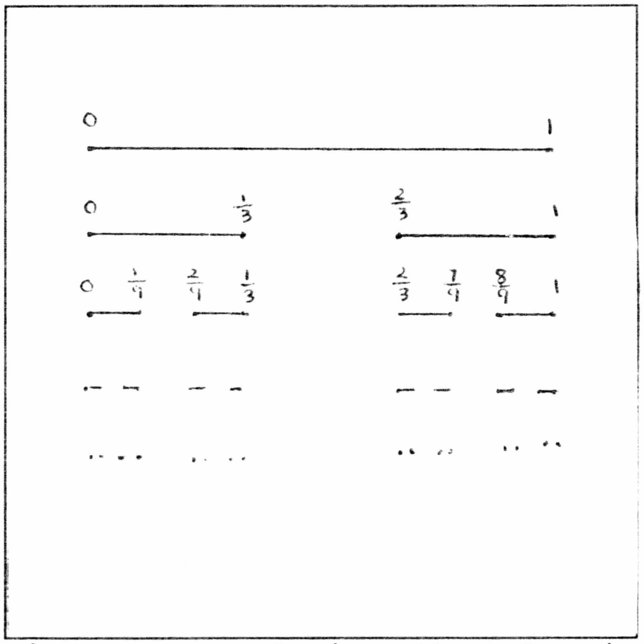

Figure 3: Producing the classic Cantor set

Consider the polynomial $p(z) = z^2 + 3$. The forward orbit of the critical point is $O^+(0) = (0,3,12,147,...)$. Since this orbit tends to increase without bound, J(p) must be a Cantor set. The graph of this Julia set, provided in the Appendix B, shows four clusters of points. Enlarging one of the clusters reveals a graph similar to the original picture and provided on the following page of the appendix. Theoretically, the Julia set has the same structure at all levels of magnification. All Julia sets share this quality, and the fact that J(p) is locally eventually onto affirms this observation: the structure of the entire Julia set derives from an arbitrarily small piece of the set.

For the second class we provide the following definition.

Definition 6.3: A curve is said to be a **Jordan curve** if it is closed and simple.

Basically, a simple curve does not intersect or overlap itself. The Julia set of $p(z) = z^2 - 0.5$, pointed out in a previous example, provides a good example of a Jordan curve.

Now consider the polynomial $p(z) = z^2 - 1$. The orbit of the critical point is $O^+(0) = (0,-1,0,-1,...)$, which has period two. Since $L(0) = 0$, zero is an attracting point, and hence, J(p) is the closure of one, two, or infinitely many Jordan curves. The graph in Appendix B indicates that J(p) consists of infinitely many Jordan curves. Notice that each Jordan curve engulfs a component of the stable set. Interestingly, points in each component eventually map into the center component and approach the orbit of the critical point; i.e., points in S(p) which lie inside a Jordan curve are part of the basin of attraction of the critical point.

Finally, we define the third class of Julia sets.

Definition 6.4: Any subset of $\mathbf{C}$ which is closed and connected and does not bound a region in $\mathbf{C}$ is called a **dendrite.**

Such sets are named for their resemblance to strips of minerals found in mines and to certain neural cells, which bear the same name. What dendrites are not named for are lightning bolts, which they also resemble.

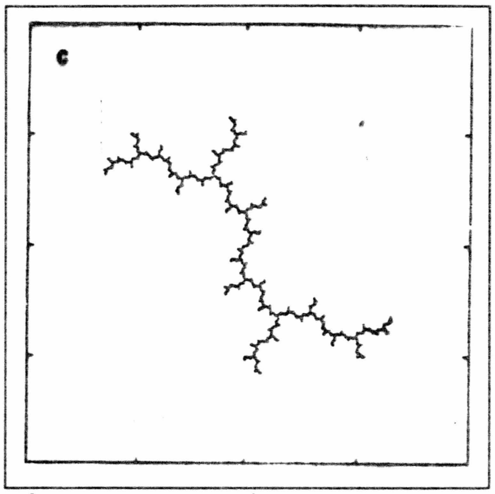

Figure 4: Julia set of $p(z) = z^2 + i$ (from <u>The Science of Fractal Images</u>)

Consider the polynomial $p(z) = z^2 + i$. The forward orbit of the critical point is $O^+(0) = (0, i, -1 + i, -i, -1 + i, -i, \ldots)$. Since the orbit of the critical point is eventually periodic but not periodic, $J(p)$ is a dendrite, as it appears in Figure 4.

7. THE MANDELBROT SET

In considering the family of functions $\{p: \mathbf{C}^* \to \mathbf{C}^* \mid p(z) = z^2 + c,\ c \in \mathbf{C}\}$, the parameter c has thus far been a fixed point in the complex plane. Allowing c to vary provides us with another way to study this family of quadratic polynomials. Consider the set of all possible values for c, namely the complex plane, which is also referred to as a parameter space. If we wish to study a certain property of Julia sets, say connectedness, we could plot the point of each value of c for which the given property holds for the Julia set of the corresponding quadratic polynomial. The resulting set of points, whose graph is provided in Appendix C, is easily defined as follows.

Definition 7.1: The **Mandelbrot set** for the family of functions $\{p: \mathbf{C}^* \rightarrow \mathbf{C}^* \mid p(z) = z^2 + c,\ c \in \mathbf{C}\}$ is $\{c \in \mathbf{C} \mid J(p)$ is connected$\}$.

Benoit Mandelbrot used computers to investigate the properties of this set while men such as Douady, Hubbard, and Sullivan proved theorems regarding the Mandelbrot set, the stable set, and more. The more recent work of these men builds upon the older work of Julia and Fatou. One must make great strides in understanding simply to keep up with the current work of such men. The field remained relatively quiet, however, during the years between the 1910's and the 1960's. The cause of the more recent interest in the subject arose with the advent of computers to speed up the repetitive iterations and to provide colorful graphs to inspire otherwise disinterested intellectuals.

Returning to the subject at hand, we present a theorem proved by Douady and Hubbard concerning the Mandelbrot set itself.

Theorem 7.2: The Mandelbrot set is connected.

Thus, a certain similarity exists between the Mandelbrot set and the Julia sets whose parameters lie within the Mandelbrot set. Indeed as Michael Barnsley points out in Fractals Everywhere, Julia sets even seem to share a similar structure with the regions of the Mandelbrot set in which their corresponding parameters lie.

It is interesting to observe the changes in the various Julia sets as we vary the parameter c. For instance, consider the sequence of Julia sets from $c = 3$ to $c = -3$ along the real axis of the Mandelbrot set. The Julia sets begin as Cantor sets, develop into cauliflower shapes whose pieces become connected at $c = 0.25$, round out to a circle, develop an infinite number of sharp edges which eventually join to divide the stable set into an infinite number of components, flatten to form the closed interval $[-2,2]$ at $c = -2$, and finally break apart again into Cantor sets.

Julia sets whose parameters lie off the real line are equally interesting. Note the examples provided in the Appendix B. As can be seen, the Julia sets from above the real line are simple mirror images of those below the real line. Also, observe that the Julia sets far away from the Mandelbrot set tend to look more like sets of two or four points, although closer observation reveals that these are Cantor sets.

Note that the main body of the Mandelbrot set has the shape of a heart, or a cardioid, and that the main piece just to the left of this cardioid is circular. The following theorems provide the equations for these two structures within the Mandelbrot set.

Theorem 7.3: $\{c \mid p \text{ has an attracting fixed point}\}$ is bounded by a cardioid of the form

$$x = \cos\Theta[1/2 - (1/2)\cos\Theta] + 1/4$$

$$y = \sin\Theta[1/2 - (1/2)\cos\Theta].$$

Proof: Let $c \in \mathbf{C}$ be an arbitrary point such that $p(z) = z^2 + c$ has an attracting fixed point z_0. Since z_0 is a fixed point, $p(z_0) = z_0$ so that $c = z_0 - z_0^2$. Since z_0 is attracting, $0 \le |L(z_0)| < 1$ so that $0 \le |z_0| < 1/2$.

In polar form, $z_0 = r(\cos\Theta + i\sin\Theta)$, where $r = |z_0|$ and Θ is in radians. Thus,

$$\begin{aligned} c &= z_0 - z_0^2 \\ &= r(\cos\Theta + i\sin\Theta) - [r(\cos\Theta + i\sin\Theta)]^2 \\ &= \cos\Theta(r - 2r^2\cos\Theta) + r^2 + i\sin\Theta(r - 2r^2\cos\Theta). \end{aligned}$$

Rewritten in parametric form, $c = x + iy$, where

$$x = \cos\Theta(r - 2r^2\cos\Theta) + r^2 \text{ and}$$

$$y = \sin\Theta(r - 2r^2\cos\Theta).$$

The graph thus described is a translation by r^2 along the x-axis of the graph given by

$$x = \cos\Theta(r - 2r^2\cos\Theta),$$

$$y = \sin\Theta(r - 2r^2\cos\Theta).$$

The equation for this graph in polar form (see Anton, p. 722) is $\Gamma = r - 2r^2\cos\Theta$, which is the general form of limaçons and cardioids. Now the ratio $1/2r$ determines the shape. Recall that $0 \le r < 1/2$. If $r = 0$, then the graph is simply a point. If $0 < r < 1/2$, then $1/2r > 1$. Hence, if $1/2r \ge 2$, then the graph is a convex limaçon. If $1 < 1/2r < 2$, then it is a dimpled limaçon. As $1/2r \to 1$, the graph approaches the shape of a cardioid. Therefore, as $r \to 1/2$, the graph of $\{c \mid p$ has an attracting fixed point$\}$ approaches the shape of a cardioid. That is, since $0 \le r < 1/2$, $\{c \mid p$ has an attracting fixed point$\}$ is bounded by a cardioid given by

$$x = \cos\theta[1/2 - (1/2)\cos\theta] + 1/4,$$

$$y = \sin\theta[1/2 - (1/2)\cos\theta]. \qquad \text{Q.E.D.}$$

Theorem 7.4: $\{c \mid p$ has an attracting periodic point of period two, but not period one$\}$ is bounded by a circle given by $|c - (-1)| = 1/4$.

Proof: Let $c \in \mathbf{C}$ be an arbitrary point such that $p(z) = z^2 + c$ has an attracting periodic point z_0 of period two. Since z_0 is periodic, $p^2(z_0) = z_0$ so that $c = -z_0^2 + z_0$ or $c = -z_0^2 - z_0 - 1$. Since z_0 is an attracting point, $0 \le |L(z_0)| < 1$ so that $0 \le |z_0^3 + z_0c| < 1/4$.

If $c = -z_0^2 + z_0$, then the inequality $0 \le |z_0^3 + z_0c| < 1/4$ implies that $0 \le |z_0| < 1/2$. Thus, c is an element of the set $\{c \mid p$ has an attracting fixed point$\}$ and is bounded by the cardioid given in the previous theorem. If $c = -z_0^2 - z_0 - 1$, then $0 \le |z_0^3 + z_0c| < 1/4$ implies that $0 \le |-z_0^2 - z_0| < 1/4$. Since $-z_0^2 - z_0 = c + 1$, the inequality is equivalent to $0 \le |c - (-1)| < 1/4$. That is, c is any point inside a circle centered at -1 with a radius between 0 and 1/4. Therefore, $\{c \mid q$ has an attracting periodic point of period two, but not period one$\}$ is bounded by a circle given by $|c - (-1)| = 1/4$. Q.E.D.

The graphs of these boundary sets, presented in the Appendix C, mark the beginning of the Mandelbrot set.

8. FRACTAL DIMENSION

In addition to working in the parameter space of the family of functions $\{p: \mathbf{C}^* \rightarrow \mathbf{C}^* \mid p(z) = z^2 + c \text{ for some } c \in \mathbf{C}\}$, Benoit Mandelbrot also studied sets with fractional dimension, D. The dimensions of a point (D = 0), a line (D = 1), and a plane (D = 2) are familiar to us. Mandelbrot observed sets which appeared to have dimensions greater than a point, but less than a line, or greater than a line, but less than a plane. He named these sets **fractals** because of their broken appearances. As the reader can attest from his observations of the graphs, most Julia sets are fractals.

Consider a coastline, Mandelbrot suggests. At the beach with a cool drink in hand, we grow restless and decide to measure a strip of the coast. Beginning with a city block as a measuring stick, we measure the strip along the water line, accounting for as many bends as possible, and jot down the value. After another drink, we grab a smaller measuring stick, say a meter stick, and repeat the experiment. We find that the measurement is greater because we have accounted for more of the bends and rocks and so forth. Continuing to drink and measure the strip of coastline with progressively smaller measuring sticks, we find, rather Mandelbrot found, that the length of the coastline continually increases and shows no signs of settling down to some "true" length even though we show signs of settling down to sleep off a mysterious headache.

Mandelbrot claims that for structures such as coastlines, the concept of length is inappropriate because it varies with the measuring device. He suggests that dimension provides an absolute measure of a coastline, i.e. a measure, not of the distance along a coastline, but of the jaggedness of the coast. In fact, Mandelbrot claims to have developed a new geometry which considers natural images, such as clouds and trees, in terms of fractional dimension. The reader may acquaint himself with some fractal images, namely a broccoli tree, a fern, and a Sierpinski triangle, found in Appendix D. The Turbo Basic code used to generate these images is provided in Appendix A.

Although mathematicians have developed many different definitions for "dimension," Mandelbrot selected a definition which allows us to compute dimensions experimentally. Rather than present Mandelbrot's definition, we present a theorem due to Barnsley which demonstrates, in the spirit of Mandelbrot yet more simply, how to compute the fractal dimension of any fractal. We conspicuously avoid defining fractals and abandon the reader to his intuition. Barnsley, however, investigates the subject more thoroughly. We assume that the fractal is situated in the complex plane, although the theorem holds for any finite-dimensional Euclidean space.

The Box Counting Theorem 8.1: Let F be a fractal lying in the complex plane. Cover **C** by closed, just-touching square boxes of side length r^n, where $0 < r < 1$, $r \in \mathbf{R}$ and $n \in \mathbf{N}$. Let $N_n(F)$ denote the number of boxes of side length r^n which intersect F. If

$$D = \lim_{n\to\infty}[\ln(N_n(F))/\ln(1/r^n)]$$

exists, then F has fractal dimension D.

To demonstrate how this theorem works, consider the classical Cantor set, which we denote C. Using boxes of side length $1/3^n$ and counting the number of boxes needed to cover C, we find

$$N_1(C) = 2 = 2^1$$
$$N_2(C) = 4 = 2^2$$
$$N_3(C) = 8 = 2^3$$
$$\vdots$$
$$N_n(C) = 2^n$$

Thus, $D = \lim_{n\to\infty}[\ln(2^n)/\ln(3^n)] = \ln(2)/\ln(3)$. So the fractal dimension of the classical Cantor set is about 0.631.

As another example, consider the Sierpinski triangle S, constructed as follows. Draw a filled right triangle with arms of length one. Remove an inverted triangle of arms length 1/2 from the center of the filled triangle. Remove an inverted triangle of arms length 1/4 from each of the three remaining triangles. Continue this process _ad infinitum_. A representation of the resulting fractal is provided in Appendix D.

Now let us compute the fractal dimension of this Sierpinski triangle. Use boxes of side length $1/2^n$. Then the number of boxes needed to cover S are

$$N_1(S) = 3 = 3^1$$
$$N_2(S) = 9 = 3^2$$
$$N_3(S) = 27 = 3^3$$
$$\vdots$$
$$N_n(S) = 3^n$$

Hence, $D = \lim_{n\to\infty}[\ln(3^n)/\ln(2^n)] = \ln(3)/\ln(2)$. Therefore, the Sierpinski triangle has fractal dimension of about 1.585.

Julia sets also have fractal dimensions. From the graphs in Appendix B these dimensions seem to range from about zero to one, where the finer Cantor sets have dimensions closer to zero and the circle has dimension one. To know whether or not any of these or other Julia sets have dimension greater than one requires more extensive research on fractal dimension. A preliminary answer might be obtained from Barnsley, who devotes a section of his text to a method of experimentally estimating the fractal dimension of sets such Julia sets.

9. UNFINISHED BUSINESS

This paper has barely touched on the abundance of information available on the topic of Julia sets. Hopefully, we have whetted the reader's appetite for the subject so that he will investigate some of the sources listed among the references. Even the reader who is dismayed with Julia sets may find a source, perhaps on an attempted application of chaotic dynamics in science, which better suits his interests. If all else fails, the reader can flip through a copy of Peitgen and Richter's The Beauty of Fractals to view their wonderful art, which humiliates the meager graphs presented in this paper.

Finally, we point out that during our first attempts to generate Julia sets using the algorithm outlined in the text, we plotted the flowering graph provided in Appendix E. We had anticipated generating the Julia set of $p(z) = z^2$, which is simply the unit circle. The program was clearly wrong. The correct algorithm, which calculates and plots successive preimages of an initial complex value $z_0 = x_0 + iy_0$, employs the following iterations:

$$x_{n+1} = \sqrt{[(x_n + \sqrt{(x_n^2 + y_n^2)})/2]} \text{ and}$$

$$y_{n+1} = (\text{sgn } y_n) i\sqrt{[(- x_n + \sqrt{(x_n^2 + y_n^2)})/2]}.$$

The mauled version of the algorithm employed the following iterations instead:

$$x'_{n+1} = \sqrt{[(x_n + \sqrt{(x_n^2 + y_n^2)})/2]} \text{ and}$$

$$y_{n+1} = (\text{sgn } y_n) i\sqrt{[(- x_{n+1} + \sqrt{(x_{n+1}^2 + y_n^2)})/2]}.$$

The difference is subtle, but the consequences, as shown in the graphs, are remarkable. The lesson which this error teaches is that the study of Julia sets depends upon one kind of iteration, but other kinds of iteration may lead to interesting results as well. Furthermore, we have not read any works which investigate sets generated via an alternate method of iteration. We leave the reader with an open question: Is the graph in Appendix E fractal?

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APPENDIX A: TURBO BASIC PROGRAMS

The following programs were written and tested in Turbo Basic on an IBM pc with graphics capability. The algorithms and much of the code for these programs were found in Michael Barnsley's Fractals Everywhere. The first page shows a sample output, not including the graph, from the first program. The first program generates Julia sets for the family of quadratic polynomials $\{p: \mathbf{C}^* \to \mathbf{C}^* \mid p(z) = z^2 + c$, for some $c \in \mathbf{C}\}$. Many examples of the output from this program are provided in Appendix B. The last three programs generate the fractal images of a Sierpinski triangle, a fern, and a tree, respectively. These three fractals are provided in Appendix C.

```
Julia Set Generator

Would you like to read the instructions (y/n)?  y

        This program generates a Julia set for a quadratic
function of the form f(z) = z^2 + c, where z = x + yi is
a complex variable and c = a + bi is a complex constant.
When inputting complex values such as z = x + yi,  enter
the real parts, x and y, separated by a space on the same
line.  The initial value of z may be any point in the complex
plane, except a fixed point, but it helps if the initial
value is a point in the Julia set.  The plotting window
is the portion of the complex plane which you wish to
examine.  The window is determined by its lower left- and
upper right-hand points.

Enter a value for the parameter c = a + bi:  0 1000000
Enter an initial value z = x + yi:  1 1
Enter the number of points to compute:  5000
Enter the plotting window v = d + ei, u = f + gi:  -1000 -1000 1000 1000

When graph is complete, press esc to return to Turbo Basic.
Press any key to begin plotting.
```

```
'Julia Set Generator

'John W. Deighan  Washington and Lee University  3/15/89

'      This program generates Julia sets for quadratic functions
'of the form f(z) = z^2 + c, where z is a complex variable and c
'is a complex parameter which is fixed for each set.  The program
'employs a random selection process in the main loop.

cls
print "Julia Set Generator"
print
input "Would you like to read the instructions (y/n)?  ", i$
if i$ = "y" then gosub instructions
print
input "Enter a value for the parameter c = a + bi:  ", a!, b!
input "Enter an initial value z = x + yi:  ", x!, y!
input "Enter the number of points to compute:  ", n%
input "Enter the plotting window v = d + ei, u = f + gi:  ", d!,
      e!, f!, g!
print
print "When graph is complete, press esc to return to
TurboBasic."
print "Press any key to begin plotting."
while not instat
wend
screen 2
cls
window (d!,e!) - (f!,g!)
for i% = 1 to (n% + 10)
     newx! = sqr((sqr((x!-a!)*(x!-a!) + (y!-b!)*(y!-b!)) +
                 x! - a!)/2)
     if sgn(y!) = -1 then
          sign% = -1
     else
          sign% = +1
     end if
     newy! = (sign%)*sqr((sqr((x!-a!)*(x!-a!) + (y!-b!)*(y!-b!))
                          - x! + a!)/2)
     r! = rnd
     if (0 <= r!) and (r! < 0.5) then
          x! = -newx!
          y! = -newy!
     else
          x! = newx!
          y! = newy!
     end if
     if i% > 10 then pset ((0.75*x!),y!)
next
end


'continued
```

```
instructions:
     print
     print "     This program generates a Julia set for a"
     print "quadratic function of the form f(z) = z^2 + c, where"
     print "z = x + yi is a complex variable and c = a + bi is a"
     print "complex constant.  When inputting complex values"
     print "such as z = x + yi,  enter the real parts, x and y,"
     print "on the same line separated by a space.  The initial"
     print "value of z may be any point in the complex plane,
print "but it helps if the initial value is a point in the"
print "Julia set.  The plotting window is the portion of"   print
"the complex plane which you wish to examine.  The"       print
"window is determined by its lower left- and upper"
     print "right-hand points."
return
```

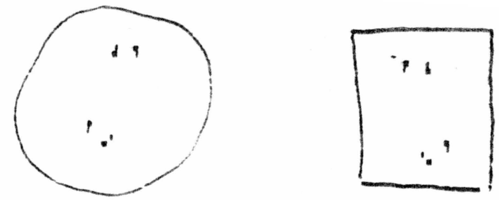
```
rem Example of Chaos Algorithm
rem Michael Barnsley, Fractals Everywhere, p. 91
rem This program generates a Sierpinski triangle.

a[1] = 0.5 : b[1] =  0 : c[1] =  0
d[1] = 0.5 : e[1] =  1 : f[1] =  1
a[2] = 0.5 : b[2] =  0 : c[2] =  0
d[2] = 0.5 : e[2] = 50 : f[2] =  1
a[3] = 0.5 : b[3] =  0 : c[3] =  0
d[3] = 0.5 : e[3] = 50 : f[3] = 50
screen 2 : cls                       'initialize computer graphics
window (0,0)-(100,100)               'set plotting window to
                                     '0 < x < 1, 0 < y < 1
x = 0 : y = 0 : numits = 20000       'initialize (x,y) and define
                                     'the  number  of  iterations,
                                     'numits
for n = 1 to numits                  'Random Iteration begins!
k = int(3*rnd-0.00001) + 1           'choose one the number 1, 2,
                                     'or 3 with equal probability

rem apply affine transformation number k to (x,y)

newx = a[k]*x + b[k]*y + e[k]
newy = c[k]*x + d[k]*y + f[k]
x = newx : y = newy                  'set (x,y) to the point thus
                                     'obtained
if n > 10 then pset(x,y)             'plot (x,y) after the first 10
                                     'iterations
next : end
```

```
rem Example of Chaos Algorithm
rem Michael Barnsley, Fractals Everywhere, p. 91
rem This program generates a fern.

a[1] =  0     : b[1] =  0     : c[1] =  0
d[1] =  0.16  : e[1] =  0     : f[1] =  0
a[2] =  0.85  : b[2] =  0.04  : c[2] = -0.04
d[2] =  0.85  : e[2] =  0     : f[2] =  1.6
a[3] =  0.2   : b[3] = -0.26  : c[3] =  0.23
d[3] =  0.22  : e[3] =  0     : f[3] =  1.6
a[4] = -0.15  : b[4] =  0.28  : c[4] =  0.26
d[4] =  0.24  : e[4] =  0     : f[4] =  0.44
screen 2 : cls                         'initialize computer graphics
window (-5,0)-(5,10)                   'set plotting window
x = 0 : y = 0 : numits = 50000         'initialize (x,y) and define
                                       'the number of iterations,
                              'numits
for n = 1 to numits                    'Random Iteration begins!
r! = rnd
if (0 < r!) and (r! <= 0.01) then k = 1
if (0.01 < r!) and (r! <= 0.86) then k = 2
if (0.86 < r!) and (r! <= 0.93) then k = 3
if (0.93 < r!) and (r! < 1) then k = 4

rem apply affine transformation number k to (x,y)

newx = a[k]*x + b[k]*y + e[k]
newy = c[k]*x + d[k]*y + f[k]
x = newx : y = newy                    'set (x,y) to the point thus
                                       'obtained
if n > 10 then pset(x,y)               'plot (x,y) after the first 10
                                       'iterations
next : end
```

```
rem Example of Chaos Algorithm
rem Michael Barnsley, Fractals Everywhere, p. 91
rem This program generates a fractal tree.

a[1] = 0    : b[1] =  0     : c[1] =  0
d[1] = 0.5  : e[1] =  0     : f[1] =  0
a[2] = 0.42 : b[2] = -0.42  : c[2] =  0.42
d[2] = 0.42 : e[2] =  0     : f[2] =  0.2
a[3] = 0.42 : b[3] =  0.42  : c[3] = -0.42
d[3] = 0.42 : e[3] =  0     : f[3] =  0.2
a[4] = 0.1  : b[4] =  0     : c[4] =  0
d[4] = 0.1  : e[4] =  0     : f[4] =  0.2
screen 2 : cls                         'initialize computer graphics
window (-0.25,-0.05)-(0.25,0.45)       'set plotting window
x = 0 : y = 0 : numits = 30000         'initialize (x,y) and define
                                       'the  number  of  iterations,
                                       'numits
for n = 1 to numits                    'Random Iteration begins!
r! = rnd
if (0 < r!) and (r! <= 0.05) then k = 1
if (0.05 < r!) and (r! <= 0.45) then k = 2
if (0.45 < r!) and (r! <= 0.85) then k = 3
if (0.85 < r!) and (r! < 1) then k = 4

rem apply affine transformation number k to (x,y)

newx = a[k]*x + b[k]*y + e[k]
newy = c[k]*x + d[k]*y + f[k]
x = newx : y = newy                    'set (x,y) to the point thus
                                   'obtained
if n > 10 then pset(x,y)               'plot (x,y) after the first 10
                                       'iterations
next : end
```

APPENDIX B: IMAGES OF JULIA SETS

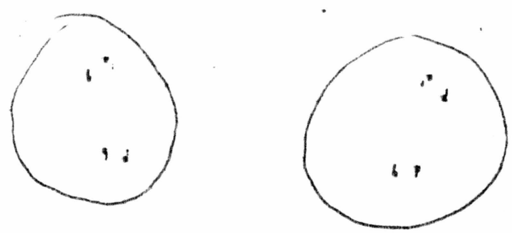
The following graphs were produced using the program provided in Appendix A. The quadratic polynomial associated with each Julia set is indicated. With the exception of the second graph, each graph shows the portion of the complex plane form -2 -2i to +2 + 2i with zero at the center of each set. The second graph, which is an enlargement of the first, has a window from 0.25 + 1.5i to 0.5 + 2i. Each Julia set is classified as outlined in Section 6, and special features of some sets are indicated.

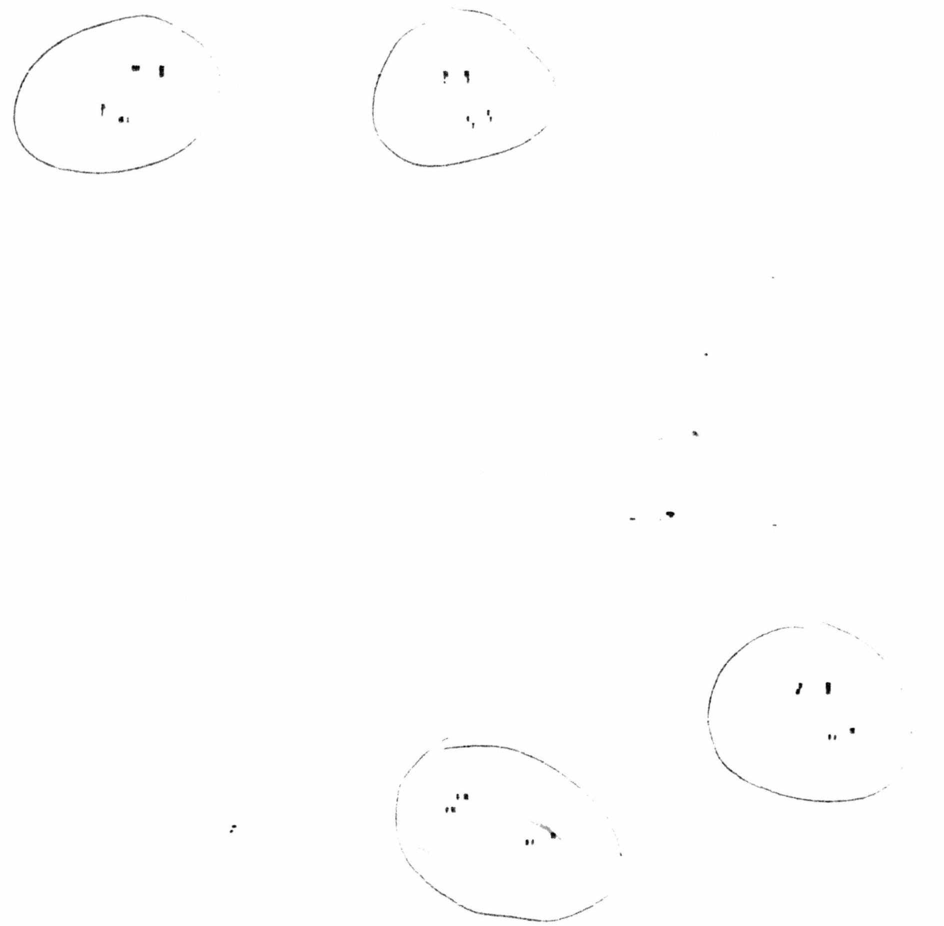


$f(z) = z^2 + 3$

$(-2,-2) - (2,2)$

Enclosed regions are part of this Cantor set. The boxed points are magnified on the following page.

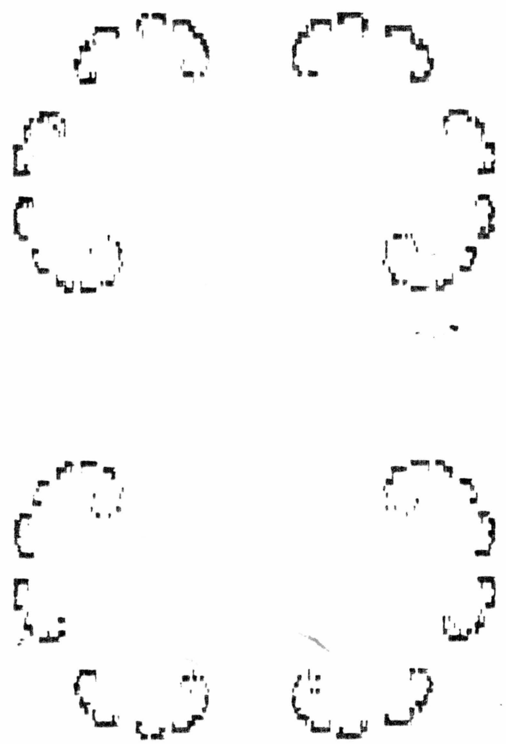




$f(z) = z^2 + 3$

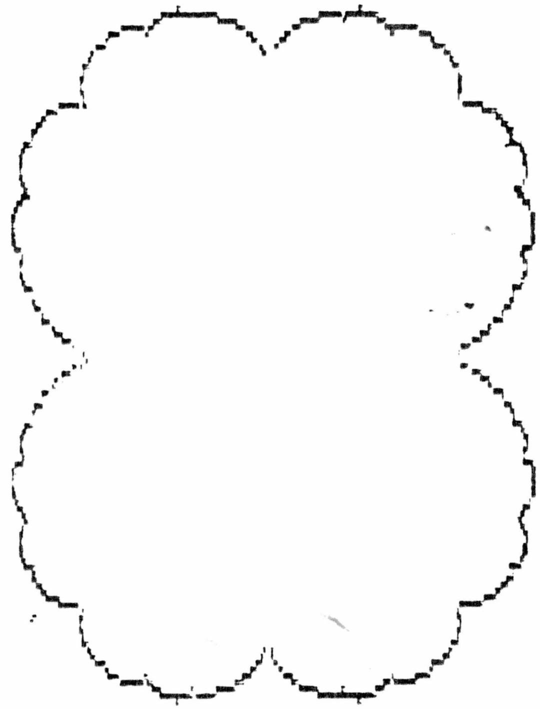
(0.25, 1.5) – (0.5, 2)

Magnification of boxed region of Cantor set on previous page. Note the similarity of structure.



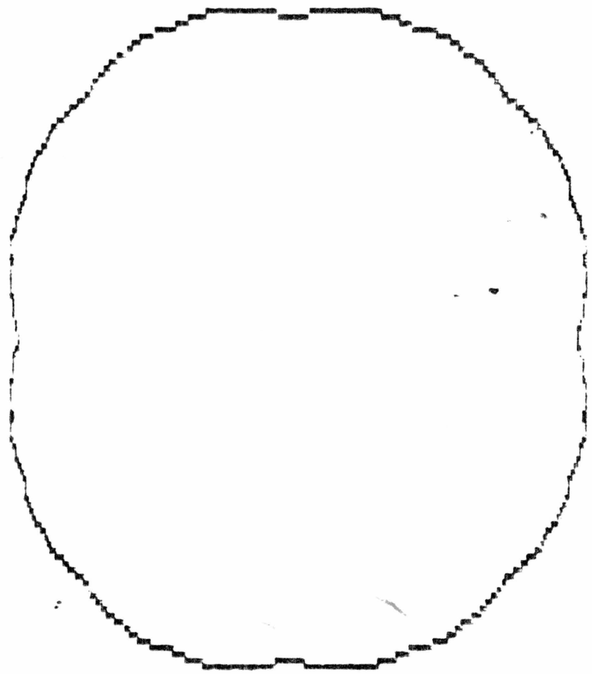
$$f(z) = z^2 + 0.4$$

"Cauliflower" Cantor set



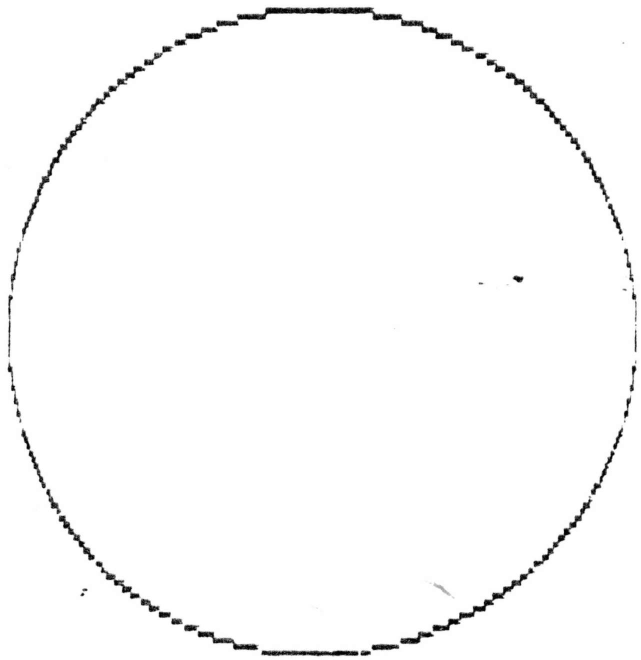
$f(z) = z^2 + 0.25$

A Jordan curve



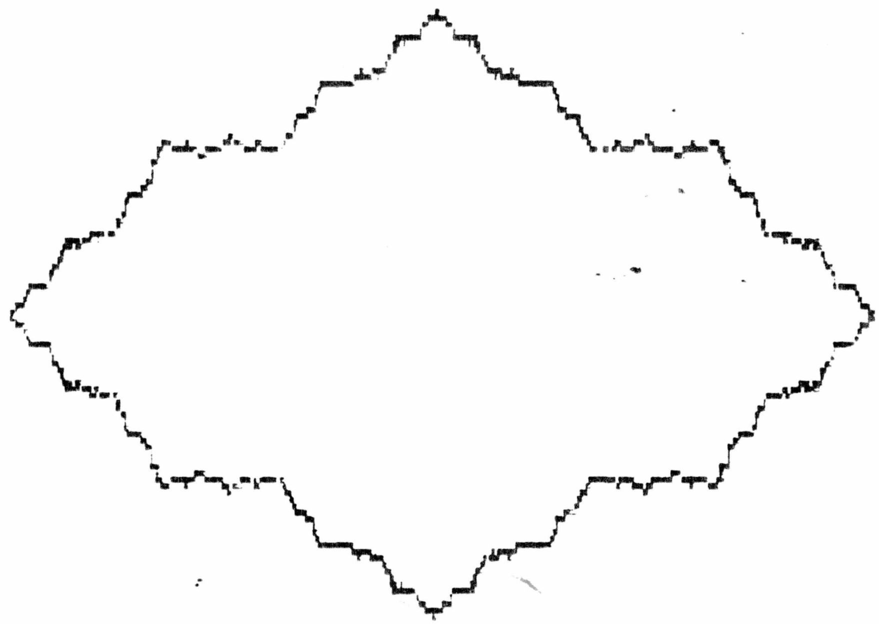
$f(z) = z^2 + 0.1$

A Jordan curve.



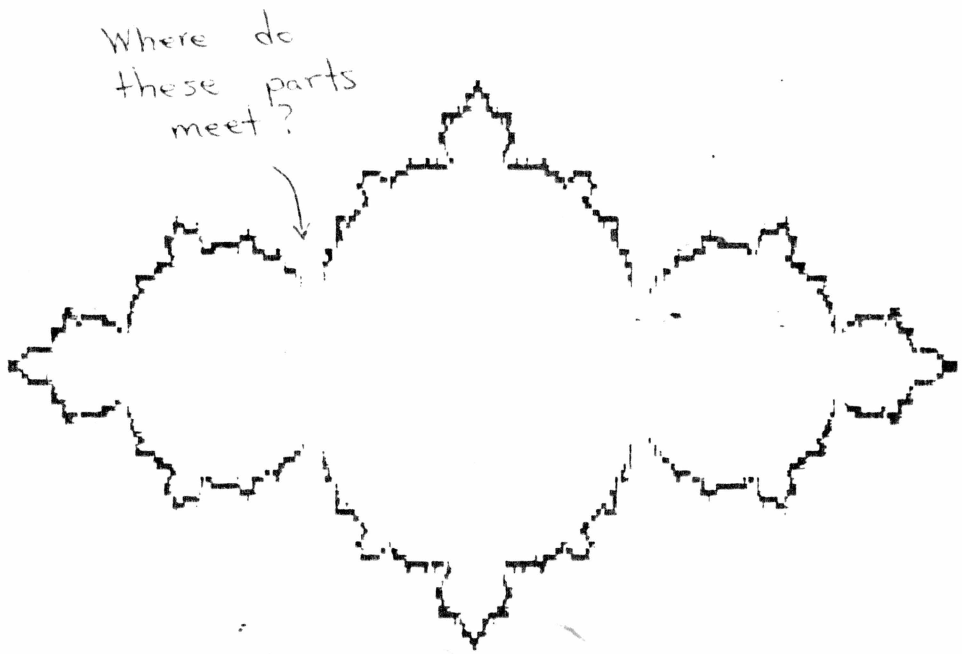
$f(z) = z^2$

The unit circle



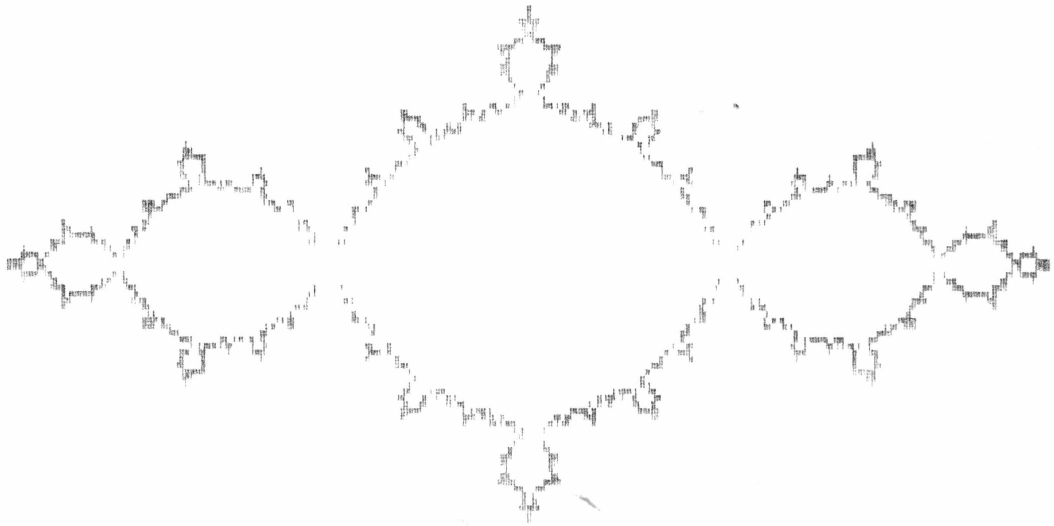
$f(z) = z^2 - 0.5$

J(f) is a Jordan curve.
The graph consists only of edges; i.e.,
J(f) is nowhere differentiable

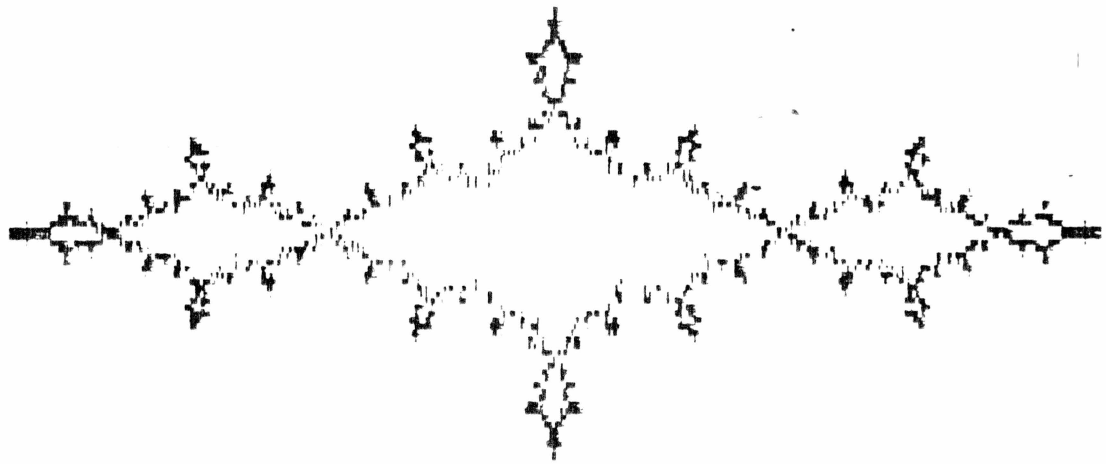


$$f(z) = z^2 - 0.75$$

Apparently, J(f) is a Jordan curve.
Note that the parameter lies between the main cardioid and circle in the Mandelbrot set.

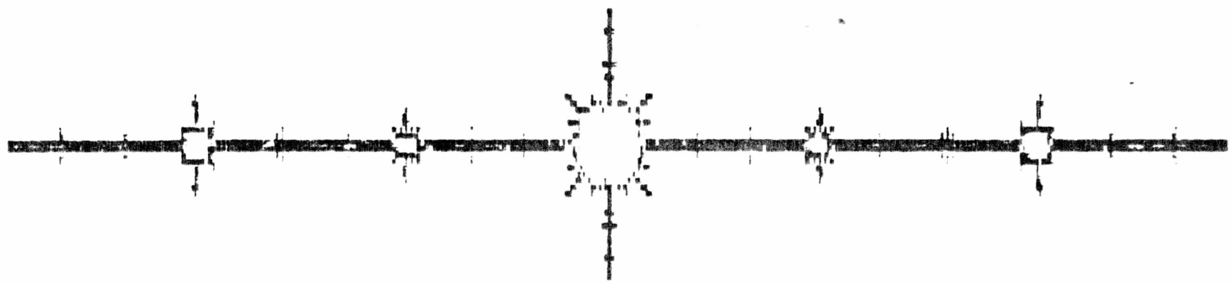


$f(z) = z^2 - 1$



$f(z) = z^2 - 1.25$

J(f) surrounds infinite number of components of S(f). Note that a greater number of iterations would have produced a sharper image.



$f(z) = z^2 - 1.75$

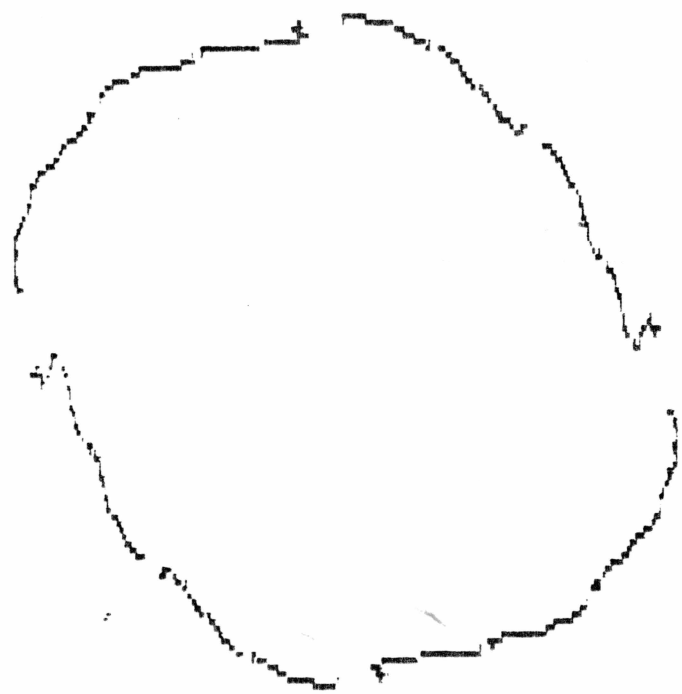
J(f) surrounds infinitely many components of S(f). Note that the horizontal line lies on the real axis and has no thickness.

$f(z) = z^2 - 2$

The real interval [-2,2]
Note that thickness of line and small "hairs" are due to pixel size and computational error, respectively.

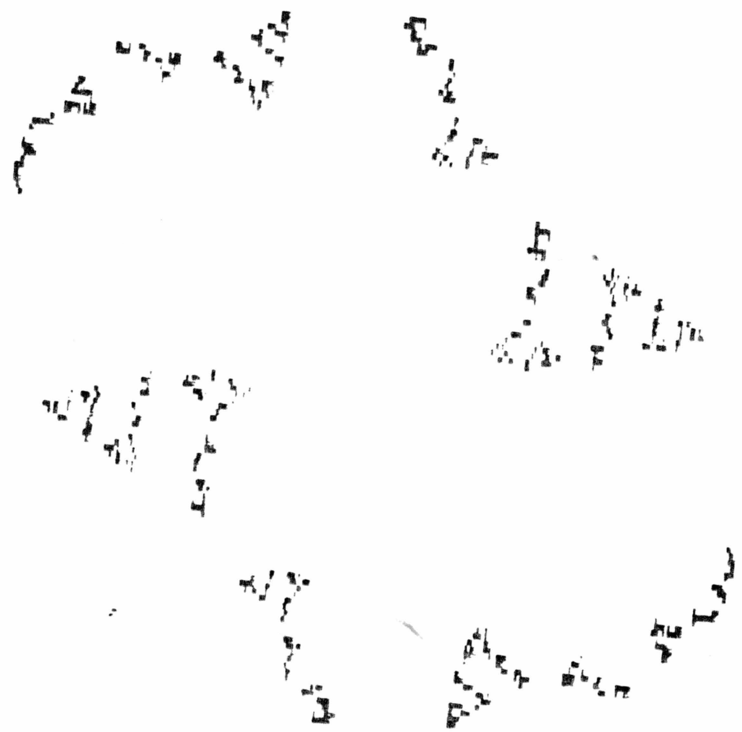
$$f(z) = z^2 - 3$$

This Cantor set lies on the real line, apparently bounded by about −2.5 below and +2.5 above.



$f(z) = z^2 + 0.2i$

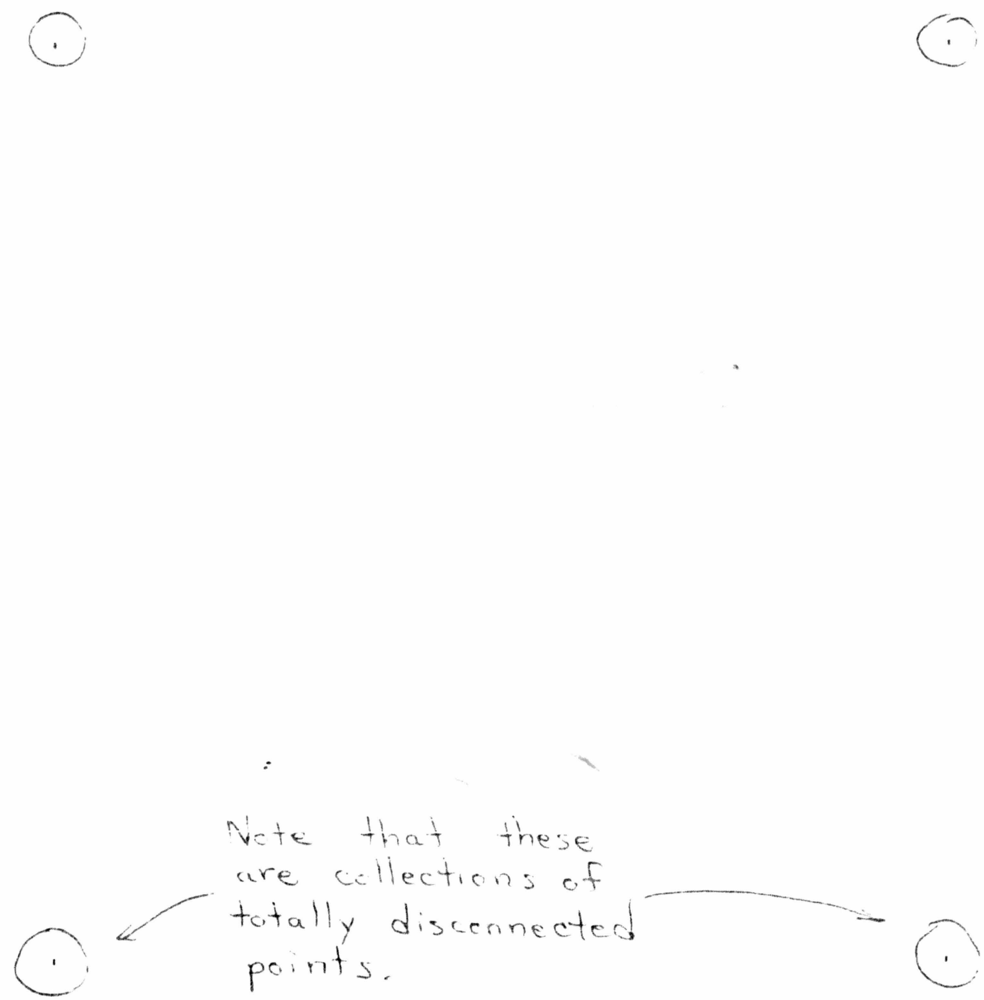
Cantor set?
Parameter lies within Mandelbrot set
so that J(f) should be connected.



$f(z) = z^2 + 0.6i$

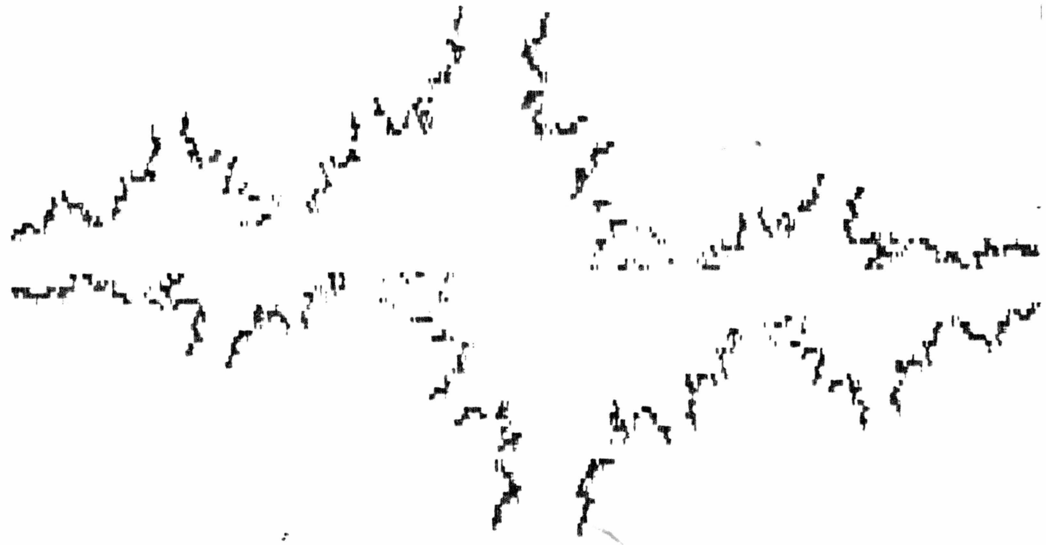
Cantor set?

Parameter lies in Mandelbrot set
so that $J(f)$ should be connected.



$$f(z) = z^2 + 1000000i$$

Circled points are in Julia set.
Window: (-1000, -1000) to (1000, 1000)
Cantor set



$f(z) = z^2 - 1 + 0.25i$

Cantor set

APPENDIX C: IMAGES OF THE MANDELBROT SET

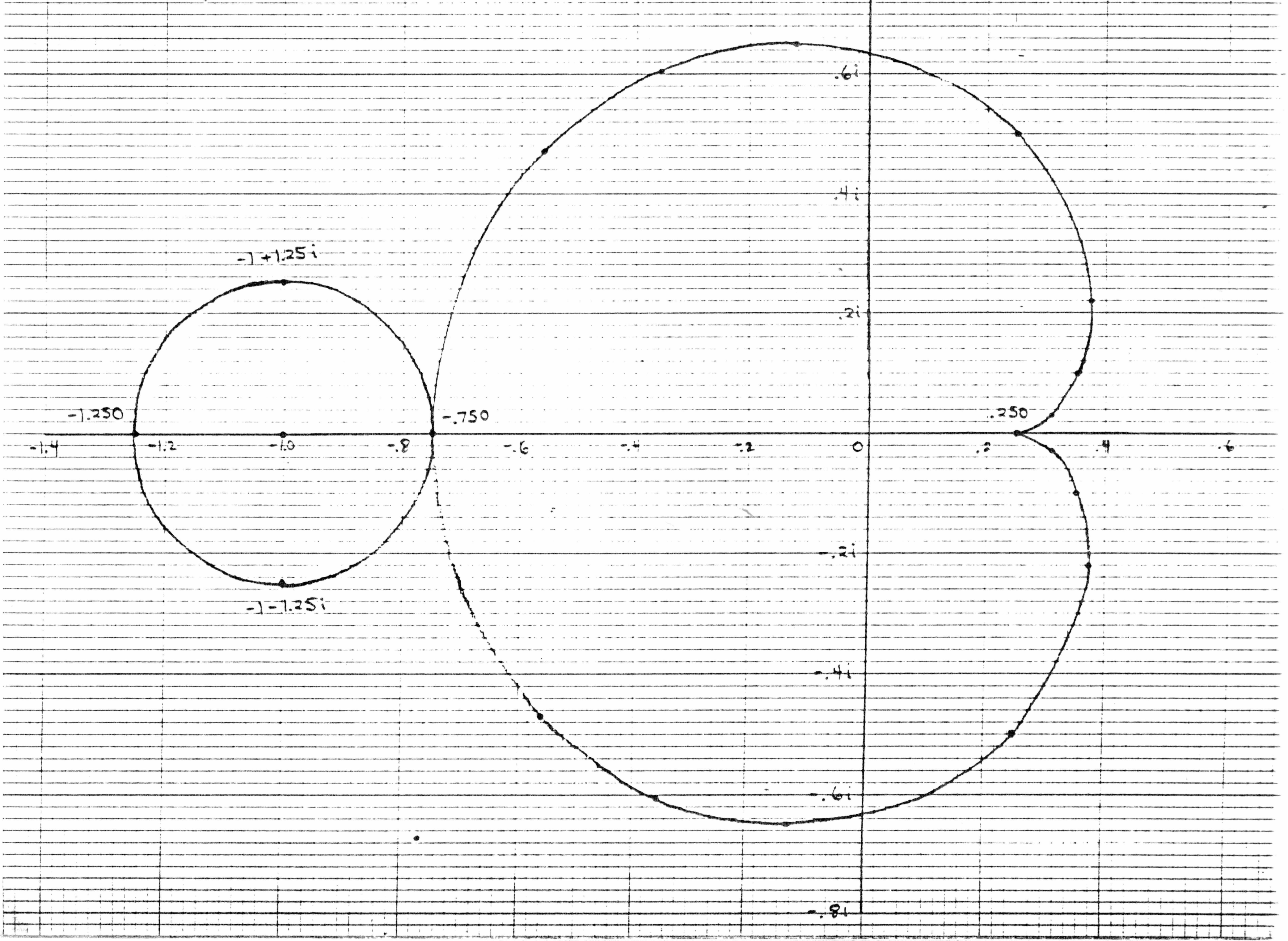
The following sketch marks the beginning of the Mandelbrot set, based on the analysis in Section 7. Points lying in the interior of the cardioid and the circle mark parameter values c for which the Julia set of $p(z) = z^2 + c$ is connected. The equation of the cardioid is

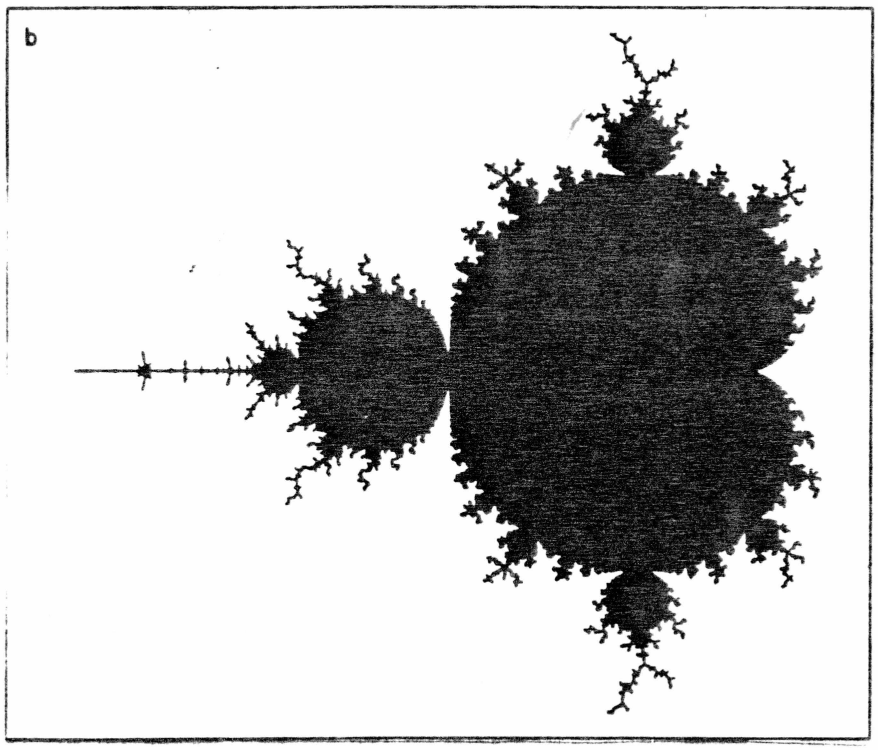
$$x = \cos\Theta[1/2 - (1/2)\cos\Theta] + 1/4$$
$$y = \sin\Theta[1/2 - (1/2)\cos\Theta].$$

The equation of the circle is $|c - (-1)| = 1/4$.

The second graph is an image of the Mandelbrot set from The Science of Fractal Images.

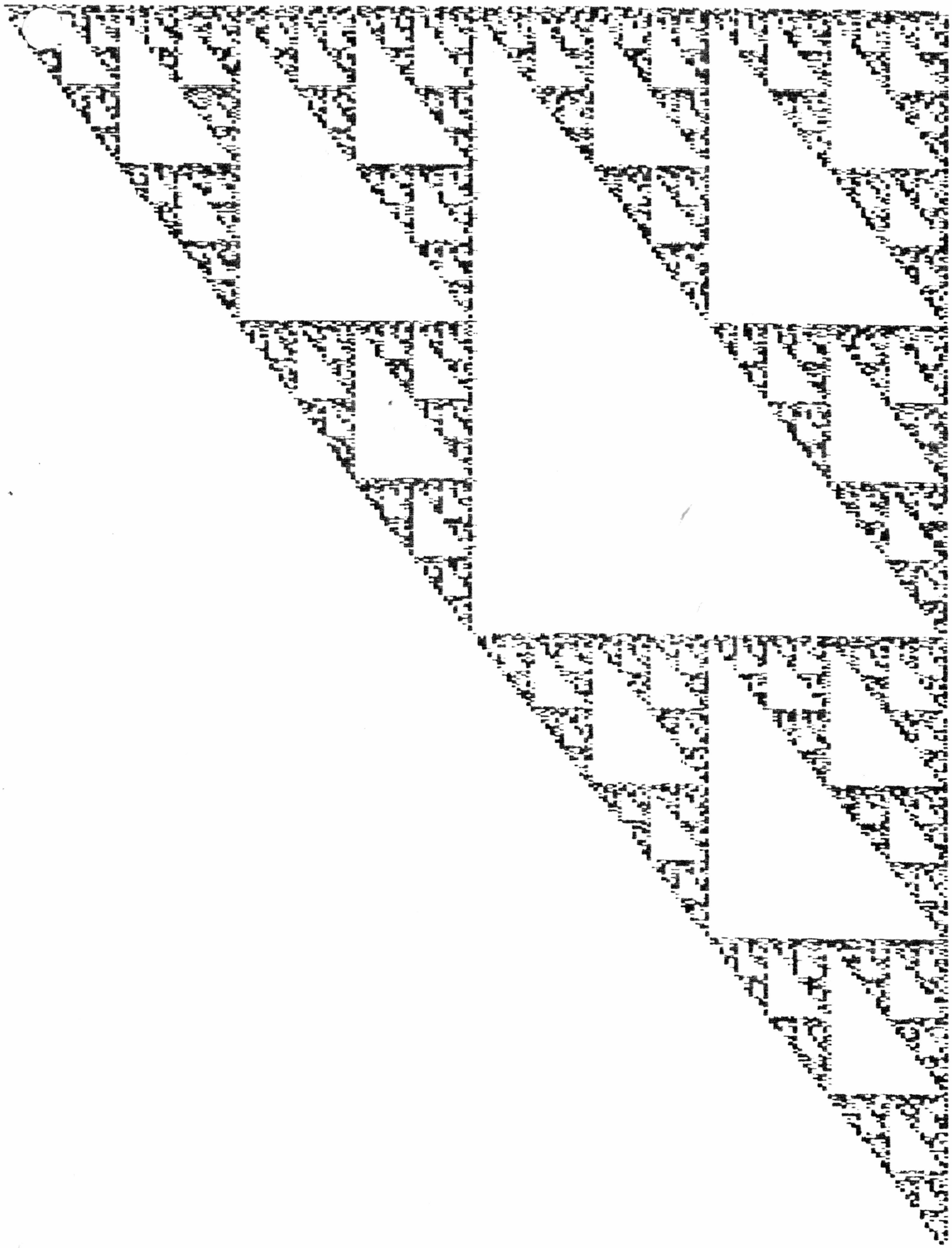
Exercise 4 : Boundary for $\{c \mid Q_c$ has an attractive periodic point of period two$\}$

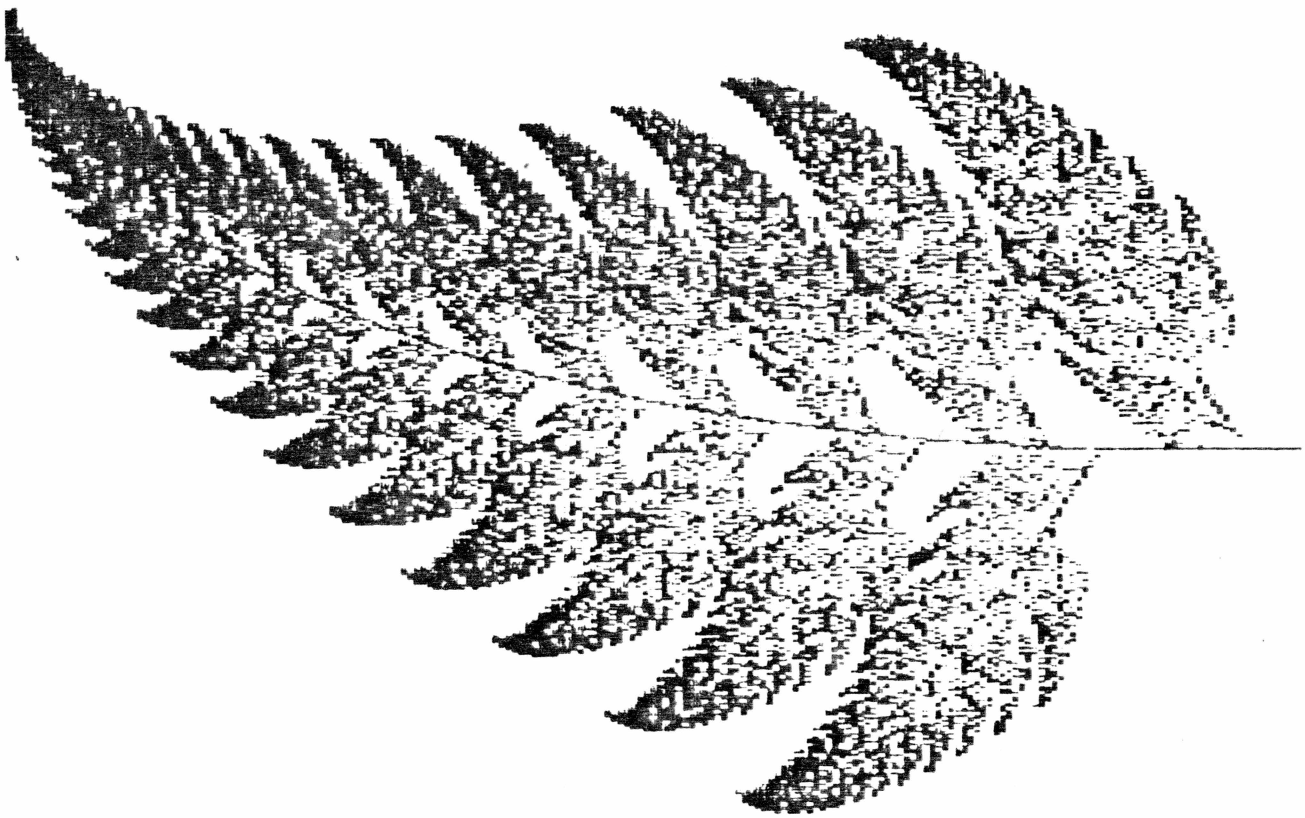


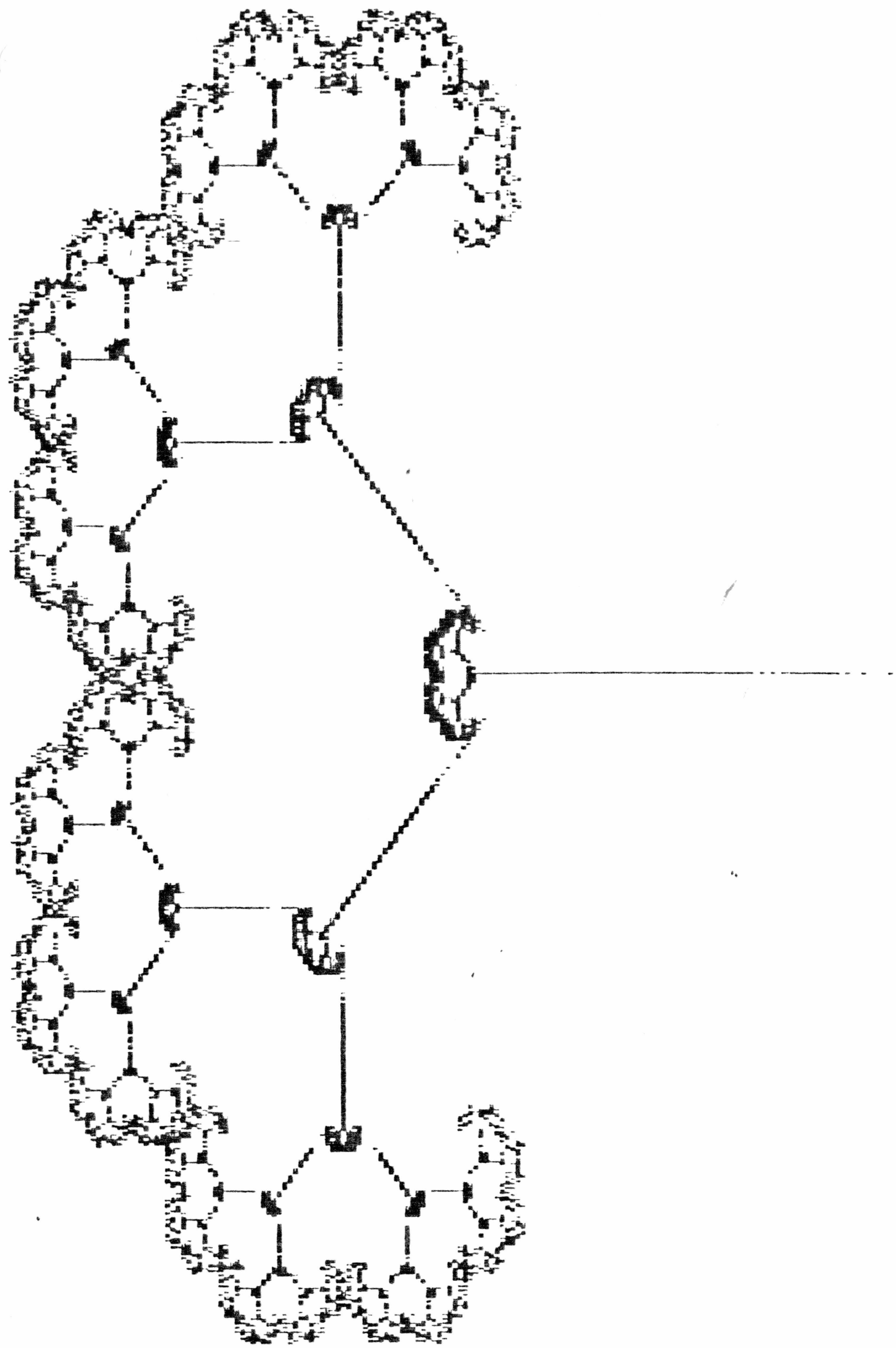


APPENDIX D: FRACTAL IMAGES

The three graphs in this appendix were produced using the programs provided in Appendix A. These fractal images are a Sierpinski triangle, a fern, and a tree.

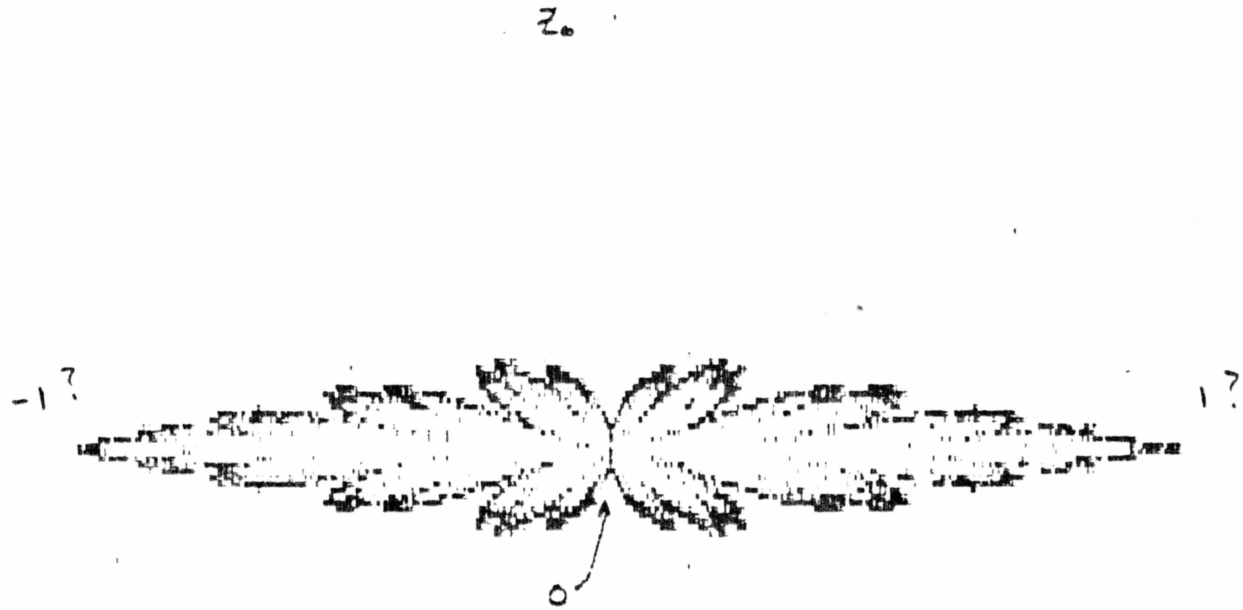






APPENDIX E: AN UNIDENTIFIED IMAGE

The graph in this appendix was produced using an erroneous version of the program for generating Julia sets listed in Appendix A. The error is analyzed in Section 9.



$f(z) = z^2$

$z_0 = i$