Differential Hyperbolic Geometry

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Introduction

In this paper, we develop two model spaces for hyperbolic geometry using differential calculus. Our approach is to first develop the Euclidean model space $\mathbf{R}^2$ and then mirror the development for hyperbolic geometry. The differential approach is advantageous because it provides a metric for each geometry. This enables us to develop the geometric isometries. A geometric *isometry* of two dimensional space X is a one-to-one function from X into X that preserves distance and preserves angles. Felix Klein in 1872 pioneered the viewpoint that a geometry is reflected by its isometries. Thus, by developing the concept of a hyperbolic metric we can find the hyperbolic isometries and hence better understand hyperbolic geometry.

We begin by presenting a historical approach to hyperbolic geometry.

Section 1: A Historical Approach to Hyperbolic Geometry

The first formal study of geometry was developed in 300 B.C. by the Greek mathematician Euclid. Developing the works of his predecessors, he set forth a thirteen volume work entitled *The Elements* , in which he developed an axiomatic approach to the study of geometry. An *axiomatic* approach bases all of its theorems on a complete list of axioms, or postulates, which are defined to be assumed truths. Euclid's geometry was based on five axioms. They were presented as such [1]:

1. For every point P and for every point Q not equal to P there exists a unique line l that passes through P and Q.

2. For every segment AB and for every segment CD there exists a unique point E such that B is between A and E and segment CD is congruent to segment BE.

3. For every point O and every point A not equal to O there exists a circle with center O and radius OA.

4. All right angles are congruent to each other.

5. For every line l and for every point P that does not lie on l there exists a unique line

m through P that is parallel to l.

The first four postulates can be easily visualized with the use of straightedges, compasses, and protractors. However, in the fifth postulate one must speculate on what happens to lines as they extend towards infinty. Because of this, mathematicians thought that it should be possible to deduce the fifth postulate from the first four postulates. This idea triggered a flurry of activity on the part of mathematicians. For 2,000 years mathematicians from Ptolemy to Legendre attempted to prove the fifth postulate, using various methods. It appeared that there must be some way to derive the postulate, yet no one could give a rigorous proof.

The independence of the fifth postulate did not surface until the beginning of the nineteenth century. At that time, three mathematicians from different parts of the world each separately contributed to the solving of the problem. Carl Freidrich Gauss of Germany (1777-1855), Nikolai Ivanovich Lobachevsky of Russia (1792-1856), and Janos Bolyai of Hungary (1802-1860) each proposed the idea that there might be no proof of the fifth postulate. They approached the subject by considering a contrapositive to the fifth postulate along with Euclid's first four postulates. In other words, since it appeared that the fifth postulate was indeed independent of the first four postulates, an entirely different geometry would arise if one assumed a different fifth postulate along with the first four of Euclid's postulates. One contrapositive to the fifth postulate is stated as such [1]:

Hyperbolic Postulate. Given a line l and a point P not on l there exist at least two lines through P parallel to l.

Assuming this postulate together with Euclid's first four postulates results in an entirely new, yet logically consistent geometry. This geometry is called hyperbolic geometry. By changing one axiom, mathematicians discovered a completely different geometry.

An extremely important point to comprehend is that hyperbolic geometry's consistencey does not negate euclidean geometry's consistency. Rather, assuming one of the geometries is consistent proves the other geometry is equally consistent. Italian mathematician Eugenio Beltrami proved that hyperbolic geometry is consistent if and only if Euclidean geometry is consistent. Furthermore, if Euclidean geometry is consistent then no proof or disproof of the parallel postulate from the rest of the postulates will ever be found. Hence attempts to prove Euclid's fifth postulate were in vain.

The concept of mutually dependent geometries allows one to understand hyperbolic geometry. It is possible to learn about hyperbolic geometry by mimicking various approaches to Euclidean geometry. Thus, a logical way to approach hyperbolic geometry is to take the terms and concepts of Euclidean geometry and translate them into hyperbolic terms. Mathematicians used this approach to axiomatically develop the subject.

Our approach will be to develop hyperbolic geometry using calculus. Although Euclid's approach is useful in the sense that the axiomatic approach is entirely "geometric" in its presentation, it lacks the flexibility that is needed to be useful in higher mathematics. In other words, developing from a differential perspective will enable us to use calculus, algebra, and group theory to analytically describe hyperbolic geometry.

Section 2: Euclidean Geometry

In order to motivate our study of hyperbolic geometry, it is useful to first understand some different approaches to Euclidean geometry. In so doing, we will be able to use our knowledge of Euclidean geometry to motivate our questioning into hyperbolic geometry.

The two approaches we will examine are the isometric approach and the differential approach. For this reason, we will introduce the isometries of the Euclidean plane. The differential approach, on the other hand, involves using calculus to describe Euclidean geometry. We will use this approach to find angle-preserving mappings and also examine lengths of curves and areas of regions in the Euclidean plane.

Isometric Approach

We first want to define a *model space* for Euclidean geometry. A *model space* is a realization of an axiomatic system. The model space for Euclidean geometry is the Euclidean plane. The Euclidean plane is defined as the set $\mathbf{R}^2 = \{(x,y)|x,y \in \mathbf{R}\}$. The Euclidean metric, i.e. the infinitesimal distance function, is defined as $ds^2 = dx^2 + dy^2$. Euclidean distance $d(P_1, P_2)$ between points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is defined to be

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Since $\mathbf{R}^2$ is a model space, it realizes the ideas of Euclidean straight lines, angles, and circles, and is consistent with Euclid's five postulates. We assume the basic notions of Euclidean points, lines, angles and circles in analytic geometry.

One type of geometric isometry is a Euclidean isometry. A Euclidean isometry is a function $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ such that $d(f(P_1), f(P_2)) = d(P_1, P_2)$ for all $P_1, P_2 \in \mathbf{R}^2$. In other words, it preserves Euclidean distance. A unique feature of $\mathbf{R}^2$ is that distance-preserving functions are also angle-preserving functions. This is because if distance is preserved, then dot product, and hence angle measure, is also preserved.

It turns out that the isometries of the Euclidean plane form a group. This allows us to describe Euclidean geometry in terms of its isometries. The fundamental examples of Euclidean plane isometries are translations, reflections, and rotations. They are described as follows: (In each of the examples let f be defined by giving the coordinates x', y' of $f(P)$ in terms of the coordinates x, y of P.)

Example 1. *Translation $t_{(\alpha,\beta)}$ of O to (α, β)*

$$x' = \alpha + x,$$

$$y' = \beta + y.$$

Example 2. *Rotation r_θ about the origin O through angle θ.*

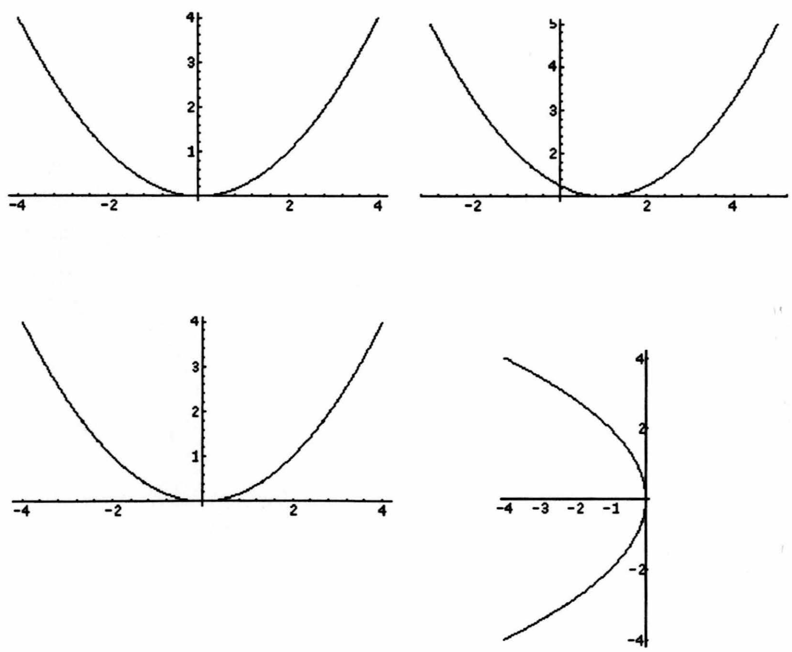
$$x' = x\cos\theta - y\sin\theta,$$

$$y' = x\sin\theta + y\cos\theta$$

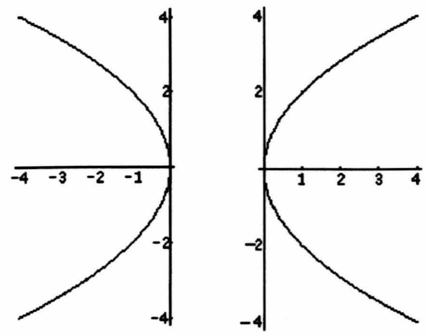
Example 3. *Reflection $\bar{r}$ (in the x-axis)*

$$x' = x,$$

$$y' = -y.$$



Example of a Euclidean translation (top) and rotation (bottom)



Example of a Euclidean reflection

Remark. Note that the origin is not unique as a point of rotation. This is because an arbitrary point (a, b) can be carried to the origin by an isometry $t^{-1}_{(a,b)}$. It is possible to rotate $\mathbf{R}^2$ about (a, b) by means of conjugation. In other words, we can translate, (a, b) to the origin, perform the indicated rotation, and then translate the origin back to (a, b). Thus, rotation of $\mathbf{R}^2$ about any point (a, b) through angle θ exists and is the isometry $t_{(a,b)} r_\theta t^{-1}_{(a,b)}$. Hence, from the point of view of isometries, the Euclidean plane looks the same at each of its points. Also, note that reflection is possible in any line L by conjugating $\bar{r}$ with an isometry f which carries the $x-$axis to L. If L passes through the origin, let f be the rotation which takes the x-axis to L. If not, let f be the *product* of a translation and a rotation ($f = r_\theta t$). If L crosses the x-axis then let $t = t_{(x,0)}$, and if L crosses the y-axis, let $t = t_{(0,y)}$.

It is easy to check that these fundamental isometries do indeed preserve distance. Translations and reflections in the axes are obviously invariant with respect to the square of the distance $(x_2 - x_1)^2 + (y_2 - y_1)^2$. Rotations about the origin also leave the square of the distance invariant as shown below:

$$\begin{aligned}(x_2' - x_1')^2 + (y_2' - y_1')^2 &= (x_2 \cos\theta - y_2 \sin\theta - x_1 \cos\theta + y_1 \sin\theta)^2 \\ &+ (x_2 \sin\theta - y_2 \cos\theta - x_1 \sin\theta + y_1 \cos\theta)^2\end{aligned}$$

$$= x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Thus, arbitrary reflections leave distance invariant.

We now consider all possible isometries expressible as products of $t_{(\alpha,\beta)}$, $\bar{r}$, and r_θ. We characterize this list of isometries as the isometries *generated by* $t_{(\alpha,\beta)}$, $\bar{r}$, and r_θ.

Theorem. *The isometries generated by* $t_{(\alpha,\beta)}$, $\bar{r}$, *and* r_θ *form a group under substitution.*

Proof. It is true that each of the three examples have inverses which are also isometries. For example, $t^{-1}_{(\alpha,\beta)} = t_{(-\alpha,-\beta)}$, $\bar{r}^{-1} = \bar{r}$, and $r_\theta^{-1} = r_{-\theta}$ (since $r_\theta r_\phi = r_{\theta+\phi}$ and $r_0 = Id$). The product of isometries is associative and the identity function is an isometry so it follows that the isometries generated by $t_{(\alpha,\beta)}$, $\bar{r}$, and r_θ form a group under substitution.∎

Below we will show that all isometries of $\mathbf{R}^2$ are generated by $t_{(\alpha,\beta)}$, $\bar{r}$, and r_θ and thus the isometries of $\mathbf{R}^2$ form a group. We do this by showing that both translations and rotations are products of reflections.

Theorem. *Any translation or rotation is the product of two reflections. Conversely, the product of two reflections is a rotation or translation.*

Proof. We first show that the translation $t_{(0,\delta)}$ is the product of the reflection $\bar{r}$ in the line $y = 0$ and the refection $t_{(0,\delta/2)}\bar{r}t^{-1}_{(0,\delta/2)}$ in the line $y = \delta/2$. This is shown by the following:

$$\begin{aligned}
(x,y) &\mapsto (x,-y) \quad \text{by} \quad \bar{r} \\
&\mapsto (x,-y-\delta/2) \quad \text{by} \quad t^{-1}_{(0,\delta/2)} \\
&\mapsto (x,y+\delta/2) \quad \text{by} \quad \bar{r} \\
&\mapsto (x,y+\delta) \quad \text{by} \quad t_{(0,\delta/2)}.
\end{aligned}$$

Hence, $t_{(0,\delta/2)}\,\bar{r}\,t^{-1}_{(0,\delta/2)} \cdot \bar{r}(x,y) = t_{(0,\delta)}(x,y)$.

We next show $r_\theta = r_{\theta/2}\,\bar{r}\,r_{\theta/2}^{-1}\bar{r}(x,y)$. In other words, a rotation is the product of a reflection in the x-axis and a reflection in the line obtained by rotating the x-axis through the angle $\theta/2$. This is shown by the following:

$$
\begin{aligned}
(x,y) &\mapsto (x,-y) \quad \text{by} \quad \bar{r} \\
&\mapsto (x\cos(\theta/2) - y\sin(\theta/2), -x\sin(\theta/2) - y\cos(\theta/2)) \quad \text{by} \quad r_{\theta/2}^{-1} \\
&\mapsto (x\cos(\theta/2) - y\sin(\theta/2), -x\sin(\theta/2) + y\cos(\theta/2)) \quad \text{by} \quad \bar{r} \\
&\mapsto (x\cos^2(\theta/2) - y\sin(\theta/2)\cos(\theta/2) - x\sin^2(\theta/2) - y\sin(\theta/2)\cos(\theta/2), \\
&\qquad x\sin(\theta/2)\cos(\theta/2) - y\sin^2(\theta/2) + x\sin(\theta/2)\cos(\theta/2) + y\cos(\theta/2)) \\
&= (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) \quad \text{by} \quad r_{\theta/2}.
\end{aligned}
$$

Conversely, suppose we have reflections $\bar{r}_L, \bar{r}_M$ in lines L, M. We choose L to be the x-axis, so $\bar{r}_L = \bar{r}$, and if L meets M, we choose their intersection to be the origin. In this case, $\bar{r}_M\bar{r}_L = r_\theta$ by the previous calculation. (Let $\theta/2$ be the angle between L and M.) If L does not intersect M, then $\bar{r}_M\bar{r}_L = t_{(0,\delta)}$ by the first calculation above. ∎

Having shown that the fundamental isometries are products of reflections, we can conjecture the following theorem. We do not prove it here because the essential ideas are characterized by the result. It is however rigorously proved in Stillwell (p.10).

Theorem. *Any isometry f of $\mathbf{R}^2$ is the product of one, two, or three reflections.*

Corollary. *The isometries of $\mathbf{R}^2$ form a group.*

Proof. It is true that associativity holds for the products of isometries since isometries are maps. It is also clear that the identity map is the identity isometry. By the previous theorem, we know that all isometries are products of reflections. Since reflection in any line is self-inverse, the inverse of the isometry $\bar{r}_{L_1}\cdots\bar{r}_{L_N}$ is just $\bar{r}_{L_N}\cdots\bar{r}_{L_1}$. Thus, the isometries of $\mathbf{R}^2$ form a group. ∎

Orientation

Orientation refers to the sense of transversal of a clockwise oriented circle in $\mathbf{R}^2$. It is intuitively clear that the product of an even number of reflections preserves the sense of a clockwise oriented circle in $\mathbf{R}^2$. Such a transformation is called orientation preserving. It is also intuitively clear that the product of an odd number of reflections reverses the orientation of a circle in $\mathbf{R}^2$. Such a transformation is called orientation reversing.

The orientation preserving isometries of $\mathbf{R}^2$ are rotations and translations. The orientation reversing isometries of $\mathbf{R}^2$ are called *glide reflections.* A *glide reflection* is defined as the product of a reflection with a translation in the direction of the line of reflection. In other words, we call $\bar{f}$ a glide reflection if $\bar{f} = t_{(\alpha,\beta)}\bar{r}$. An ordinary reflection is the special case of a glide reflection with trivial translation.

Theorem (Classification of Euclidean Isometries). *Each isometry of* $\mathbf{R}^2$ *is either a rotation, translation, or glide reflection.*

Hence, we have classified the group of isometries in Euclidean plane. We will want to mirror this group theory development when we approach understanding the isometries of hyperbolic geometry. Next we discuss the differential approach to Euclidean geometry.

Differential Approach

The calculus approach presents a method of finding angle-preserving maps of the Euclidean plane. Although we have already shown that in $\mathbf{R}^2$ distance-preserving maps are also angle-preserving, we will see that this is not necessarily the case in hyperbolic geometry. Hence, being able to find the angle-preserving maps is necessary. The way to do this is with a *tangent push-forward.* This concept also gives the straight lines as curves of shortest length and shows that area is a geometric invariant.

Tangent Push-Forward

Consider the neighborhood of a point $P \in \mathbf{R}^2$ and consider any curve in $\mathbf{R}^2$ through P, where a curve is defined to be a continuous function $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ from an interval $[a, b]$ into $\mathbf{R}^2$. If γ goes through P then $\gamma(c) = P$ for $c \in [a, b]$. If γ is differentiable, then $\gamma'(c)$ is a tangent vector at P. In fact, all vectors in $T_P(\mathbf{R}^2)$, the tangent plane to $\mathbf{R}^2$ at a point P can be realized as tangents to some curve. So, consider a function $f\colon \mathbf{R}^2 \to \mathbf{R}^2$. Given a curve $\gamma(t) \in \mathbf{R}^2$ the push forward of the tangent vector $\gamma'(t)$, $f_*(\gamma'(t)) \in \mathbf{R}^2$ is defined by

$$f_*(\gamma(t)) = Df(\gamma(t))(\gamma'(t)) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} \begin{bmatrix} \gamma_1' \\ \gamma_2' \end{bmatrix}$$

where $Df = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}$ is known as the Jacobian matrix.

An angle-preserving mapping is called a *conformal* mapping.

Theorem. *f is conformal if and only if* $Df = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ -\partial f_1/\partial y & \partial f_1/\partial x \end{bmatrix}$ *or*

$$Df = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_1/\partial y & -\partial f_1/\partial x \end{bmatrix}.$$

Proof. For ease of notation we set $\partial f_1/\partial x = a$ and $\partial f_1/\partial y = b$. Let $f = (f_1, f_2)$, $Df = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, and $v = (v_1, v_2)$, $\quad w = (w_1, w_2)$.

Then $f_*(v) = Df(v) = \begin{bmatrix} av_1 + bv_2 \\ -bv_1 + av_2 \end{bmatrix}$ and $f_*(w) = Df(w) = \begin{bmatrix} aw_1 + bw_2 \\ -bw_1 + aw_2 \end{bmatrix}$.

$$\begin{aligned} \text{So, } f_*(v)\bullet f_*(w) &= (av_1 + bv_2)(aw_1 + bw_2) + (-bw_1 + av_2)(-bw_1 + aw_2) \\ &= (a^2 + b^2)(v_1w_1 + v_2w_2) \\ &= (a^2 + b^2)(v_1, v_2)\bullet(w_1, w_2) \text{ and hence } f \text{ is conformal.} \end{aligned}$$

Conversely, assume $f = (f_1, f_2)$ is conformal. Then $f_*(e_1)\bullet f_*(e_2) = 0$ and if $f_*(v)\bullet f_*(w) = \lambda(v\bullet w)\,\forall\, v, w$, then $f_*(e_1)\bullet f_*(e_1) = f_*(e_2)\bullet f_*(e_2)$ and $ac + bd = 0$. Thus, $a^2 + c^2 = b^2 + d^2$.

Summary: f will be conformal if $f_*(e_1)\bullet f_*(e_1) = f_*(e_2)\bullet f_*(e_2)$ and $f_*(e_1)\bullet f_*(e_2) = 0$.

$f_*(e_1) = Df\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \partial f_1/\partial x \\ \partial f_2/\partial x \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$ $f_*(e_2) = Df\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \partial f_1/\partial y \\ \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$

This gives the equations: $ab + cd = 0$ and $a^2 + c^2 = b^2 + d^2$,

which imply $a = d$ and $c = -b$ $\quad$ or $\quad$ $a = -d$ and $b = c$.

Requiring these matrices to be conformal gives either:

$$\left.\begin{aligned} \partial f_1/\partial x &= \partial f_2/\partial y \\ \partial f_2/\partial x &= -\partial f_1/\partial y \end{aligned}\right\} \qquad (Cauchy - Riemann \quad equations)$$

or

$$\left.\begin{aligned} \partial f_1/\partial x &= \partial f_2/\partial y \\ -\partial f_2/\partial x &= \partial f_1/\partial y. \end{aligned}\right\}$$

These will be useful in our study of hyperbolic geometry.

Thus, angle preserving maps can be represented by two different types of Jacobian matrices. The two types are as follows:

$$Df_1 = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ -\partial f_1/\partial y & \partial f_1/\partial x \end{bmatrix} \qquad \text{or } Df_2 = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_1/\partial y & -\partial f_1/\partial x \end{bmatrix}$$

Note: $|det\, Df_1| > 0$ and $|det\, Df_2| < 0$

Also important to our study are the length of curves and the area of regions in the Euclidean plane.

Length

Given points a, $b \in \mathbf{R}$, consider a curve parameterized by γ in $\mathbf{R}$ such that $\gamma = \gamma_1(t) + \gamma_2(t)$ which starts at $\gamma(a)$ and end at $\gamma(b)$. Such a curve has a length if it is differentiable. The length of γ is defined by

$$\begin{aligned} L(\gamma) &= \int_a^b \sqrt{\gamma'(t) \bullet \gamma'(t)}\, dt \\ &= \int_a^b \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2}\, dt \end{aligned}$$

.

Definition. A *geodesic* between points p_1 and p_2 is a curve with endpoints p_1 and p_2 that has shortest length.

Theorem. *Euclidean geodesics are straight lines.*

The proof presented here may appear more complicated than necessary. However, this proof can be translated very easily for the case of hyperbolic geodesics.

Proof. Without loss of generality, we can consider a curve $\gamma(t) \in \mathbf{R}^2$ such that $\gamma(a) = p$ and $\gamma(b) = q$ where p, q lie on the y-axis. Then since

$$\begin{aligned} L(\gamma) &= \int_a^b \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2}\, dt \\ &\geq \int_a^b \sqrt{\gamma_1'(t)^2}\, dt \quad = \int_a^b |\gamma_1'(t)|\, dt \\ &\geq \int_a^b \gamma_1'(t)\, dt \\ &= \gamma_1(b) - \gamma_1(a) = q_1 - p_1. \end{aligned}$$

Further, if $L(\gamma) = q_1 - p_1$, then $\gamma_2'(t) = 0$ and also $\gamma_1'(t) \geq 0$. Hence, the curve of shortest Euclidean distance between two points is a line.

Area

Definition. Let D be a region in the Euclidean plane. Then the *area of* D is defined as

$$A(D) = \iint_D dA = \iint_D dx\, dy.$$

The intuitve description of calculating area depends on the notion of partitions. Given a region D we find $A(D)$ by partitioning D into infinitesimal regions whose areas are simple to find. Then the area of D is approximated by a sum of the areas of the regions. In other words, if $D_1, D_2, \cdots, D_n$ form a partition of D then the area of D is the sum of the areas of $D_1, D_2, \cdots, D_n$.

Theorem. *Length and area are invariant under Euclidean isometries.*

Proof. It is easy to show length is invariant under isometries. This is because $f_*(\gamma') \bullet f_*(\gamma') = \gamma' \bullet \gamma'$. To show area is invariant, we show

$$A(f(D)) = \iint_{f(D)} dx dy = \iint_D |\det(Df)| dx dy = A(D).$$

We have seen that the fundamental isometries for $\mathbf{R}^2$ are translations, rotations, and reflections. Each of the fundamental Euclidean isometries can be associated to a matrix. They are as follows:

$$Translation = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \; Rotation = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \; Reflection = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easily seen that each of these matrices has $|\det| = 1$. Hence $A(f(D)) = A(D)$. ∎

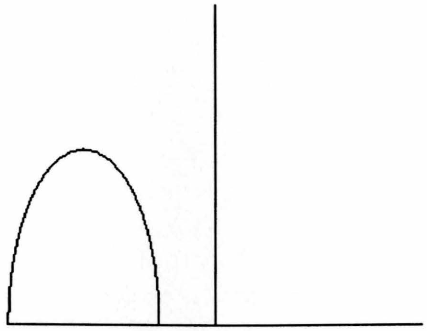
We now have a solid foundation for Euclidean geometry – the Euclidean plane, Euclidean distance, lengths of curves, and areas of regions. Our next step is to introduce hyperbolic geometry using Euclidean geometry as a guide.

Section 3: Hyperbolic Geometry

Using our knowledge of Euclidean geometry, we wish to examine hyperbolic geometry. We will develop two model spaces for hyperbolic geometry in this section; the upper-half plane model space and the unit disc model space. The fact that hyperbolic geometry has more than one model space is a very useful feature and one that distinguishes it from Euclidean geometry (which only has $\mathbf{R}^2$ as its model space.) Each of the hyperbolic model spaces realizes the hyperbolic axioms yet is unique for its own reasons. We will juxtapose both models to geometrically understand hyperbolic isometries.

Recall the Euclidean model space is the Euclidean plane $\mathbf{R}^2$. A reasonable hyperbolic model space, then, should be an analogue to the Euclidean plane that realizes the ideas of hyperbolic lines, angles, and circles, and is consistent with the hyperbolic postulates. We first introduce the upper-half plane model for hyperbolic geometry.

We define one hyperbolic model space to be the upper half plane of the complex plane with a hyperbolic distance function. We consider the real plane $\mathbf{R}^2$ as the complex plane $\mathbf{C}$ by identifying the point $(x, y) \in \mathbf{R}^2$ with $z = x + iy \in \mathbf{C}$. The upper half plane model, $\mathbf{H}^2$, is the set of points $\mathbf{H}^2 = \{z = x + iy | y > 0\}$. The upper half plane is also called the hyperbolic plane. The hyperbolic metric is defined by $ds = \sqrt{(dx^2 + dy^2)}/y$. We will see that this change in the defintion of ds causes our notions of "lines" and "circles" from Euclidean geometry to change but does not affect our notion of "angles." We will show that $\mathbf{H}^2$-lines are of two types: Euclidean semicircles lying in $\mathbf{H}^2$ with centers on the x-axis, and vertical Euclidean lines lying in $\mathbf{H}^2$. Hyperbolic circles are defined as the set of points equidistant (hyperbolically) from a point. This is the same as the Euclidean circle definition. However, with the change in distance, the center of a hyperbolic circle is not the same as the center of a Euclidean circle. The locus of points, though, is a Euclidean circle. Since distance in $\mathbf{H}^2$ is more complicated than distance in $\mathbf{R}^2$, we develop the model space for hyperbolic geometry using differential calculus. We then use Klein's approach to understand the geometry by examining its isometries. First, we develop our understanding of complex functions which will be used as tools in the differential approach.



Two Types of $\mathbf{H}^2$-Lines

Complex functions

Complex functions turn out to be useful in the study of hyperbolic geometry. Recall the identification of $(x, y) \in \mathbf{R}^2$ with $z = x + iy \in \mathbf{C}$. This identification is useful because two real variables are associated with one complex variable simplifiying computations. The complex numbers $\mathbf{C}$ support a multiplication with the usual rules except that $i^2 = -1$.

Definition.

i. $x = \frac{z+\bar{z}}{2} = \operatorname{Re} z \qquad y = \frac{z-\bar{z}}{2i} = \operatorname{Im} z$

ii. $\bar{z} = x - iy$ is the *conjugate* of z.

iii. $|z| = \sqrt{x^2 + y^2}$ is the *modulus* of z.

Let $z = (x+iy)$ and $w = (u+iv)$ be complex numbers. Then z and w obey the following properties:

i. $\dfrac{z}{w} = \dfrac{z\bar{w}}{|w|^2}$.

ii. $zw = (x + iy)(u + iv) = xu + iuy + ixv + i^2vy = (xu - vy) + (xv + uy)i$

iii. $\overline{zw} = (\bar{z})(\bar{w})$

iv. $|zw| = |z||w|$

v. $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$.

We have claimed that hyperbolic angles are the same as Euclidean angles. We are therefore interested in understanding conformal mappings of the hyperbolic plane. Recall the proof of conformal mappings in the Euclidean section. There, we introduced the Cauchy-Riemann equations for $f = f_1 + if_2$:

$$\partial f_1/\partial x = \partial f_2/\partial y$$

$$\partial f_2/\partial x = -\partial f_1/\partial y$$

These are very special in hyperbolic geometry.

Definition. A complex-valued function $f = f_1 + if_2$ is said to be *holomorphic* if $\partial f/\partial x = \frac{1}{i}\partial f/\partial y$. f is *antiholomorphic* if $\bar{f}$ is holomorphic. We note that holomorphic functions are orientation-preserving mappings of $\mathbf{H}^2$ and antiholomorphic functions are orientation-reversing mappings of $\mathbf{H}^2$.

Theorem. *$f: \mathbf{C} \to \mathbf{C}$ is holomorphic if and only if $\lim_{h\to 0} \dfrac{f(z+h)-f(z)}{h}$, $h \in \mathbf{C}$ exists.*

Proof. Assume that f is holomorphic. Then

$$\begin{aligned} f(z+h) &= f(z) + f'(z)(h) + E \\ \frac{f(z+h)-f(z)}{h} &= \frac{f'(z)+E}{h} \end{aligned}$$

Since the Taylor series for f can be written as:

$$f(z+h) = f(z) + \frac{\partial f}{\partial x}(z)h_1 + \frac{\partial f}{\partial y}(z)h_2 + E$$

where $\lim_{h\to 0}\dfrac{E}{|h|} = 0$. Hence, $\lim_{h\to 0}\dfrac{f(z+h)-f(z)}{h} = f'(z) = \dfrac{\partial f}{\partial x}$

Conversely, assume that $\lim_{h\to 0}\dfrac{f(z+h)-f(z)}{h} = l$

There are two cases:

1. For $h \in \mathbf{R}$, $l = \lim_{h\to 0}\dfrac{f(x+h,y)-f(x,y)}{h} = \dfrac{\partial f}{\partial x}$
2. For $h = it$,$l = \lim_{h\to 0}\dfrac{f(x,y+t)-f(x,y)}{it} = 1/i\dfrac{\partial f}{\partial y}$

So, $l = \dfrac{\partial f}{\partial x} = \dfrac{1}{i}\dfrac{\partial f}{\partial y}$. Hence, f is holomorphic. ∎

Holomorphic functions have a well-defined derivative defined by the difference quotient in z. The notation we use when f is holomorphic is $df/dz = f_1/x + if_2/y = f'(z)$ which is analogous to the familiar notation used in ordinary calculus.

Theorem. *Let $f: \mathbf{C} \to \mathbf{C}$ be a given function. Then $f'(z)$ exists if and only if $f = f_1 + if_2$ is differentiable in the sense of real variables and at (x, y), the functions f_1, f_2 satisfy $\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}$ and $\frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}$ (the Cauchy-Riemann equations.) Thus, if $\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, \frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}$ exist, are continuous on $\mathbf{C}$ and satisfy the Cauchy-Riemann equations, then f is holomorphic on $\mathbf{C}$.*

Proof. [2] Suppose $f'(z_0)$ exists in the limit $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$, let us take the special case that $z = x + iy_0$. Then

$$\begin{aligned}\frac{f(z) - f(z_0)}{z - z_0} &= \frac{u(x, y_0) + iv(x, y_0) - u(x_0, y_0) + iv(x_0, y_0)}{x - x_0} \\ &= \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i\frac{v(x, y_0) - v(x_0, y_0)}{x - x_0.}\end{aligned}$$

As $x \to x_0$, the left side of the equation converges to the limit $f'(z_0)$. Thus, both real and imaginary parts of the right side must converge to a limit. This limit is $\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$. Thus, $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

Now let $z = x_0 + iy$. By a similar argument, we get

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

By comparing real and imaginary parts of these equations, we derive the Cauchy-Riemann equations as well as the two formulas for $f'(z_0)$.∎

Theorem. *Suppose that f and g are holomorphic on $\mathbf{C}$. Then*

i. *$af + bg$ is holomorphic on $\mathbf{C}$ and $(af + bg)'(z) = af'(z) + bg'(z)$ for any complex numbers a and b.*

ii. *fg is holomorphic on $\mathbf{C}$ and $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.*

iii. *If $g(z) \neq 0$ for all $z \in \mathbf{C}$, then f/g is holomorphic on $\mathbf{C}$ and $[f/g]'(z) = \frac{f'(z)g(z) - g'(z)f(z)}{[g(z)]^2}$.*

iv. *$g \circ f$ is holomorphic and $(g(f(z)))' = (g'(f(z))f'(z)$.*

Proof. These proofs are easily derived from calculus. Hence, we will only prove the holomorphic portion of iv.

Let $f(z) = f(x + iy) = f(x, y)$and $g(w) = g(u + iv) = g(u, v)$.

Then $g(f(z)) = g((f_1)(z) + if_2(z)) = g(f_1(z), f_2(z))$.

$$\begin{aligned}
\frac{\partial}{\partial x}(g(f(z)) &= \frac{\partial g}{\partial u}\frac{\partial f_1}{\partial x} + i\frac{\partial g}{\partial v}\frac{\partial f_2}{\partial x} \\
&= (\frac{\partial g}{\partial z}\frac{\partial f_1}{\partial x} + i\frac{\partial g}{\partial z}(\frac{\partial f_2}{\partial x}) \\
&= \frac{\partial g}{\partial z}(\frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x}) \\
\text{So, } \frac{\partial}{\partial x}(g \circ f) &= \frac{\partial g}{\partial z}\frac{\partial f}{\partial x} = g'f' \\
\text{and similarly, } \frac{\partial}{\partial y}(g \circ f) &= \frac{\partial g}{\partial z}\frac{\partial f}{\partial y} = g'if'. \\
\text{So } \frac{\partial}{\partial x}(g \circ f) &= 1/i\frac{\partial}{\partial y}(g \circ f)
\end{aligned}$$

Hence, $(g \circ f)$ is holomorphic. ∎

Recall the idea of the tangent push-forward. It now makes sense to discuss the push forward in terms of complex numbers so that we can describe the angle-preserving maps of the complex plane.

Consider a function $f: \mathbf{C} \to \mathbf{C}$. Given a vector $h = h_1 + ih_2$, $f_*(h)$ in $\mathbf{R}^2$ is defined by

$$f_*(h) = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

Letting $h = h_1 + ih_2$ for $h \in \mathbf{C}$ with $\bar{h} = h_1 - ih_2$ gives

$$\begin{aligned}
f_*(h) &= (\frac{\partial f_1}{\partial x}h_1 + \frac{\partial f_1}{\partial y}h_2, \frac{\partial f_2}{\partial x}h_1 + \frac{\partial f_2}{\partial y}h_2) \\
&= (\frac{\partial f_1}{\partial x}h_1 + \frac{\partial f_1}{\partial y}h_2) + i(\frac{\partial f_2}{\partial x}h_1 + \frac{\partial f_2}{\partial y}h_2) \\
&= (\frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x})h_1 + (\frac{\partial f_1}{\partial y} + i\frac{\partial f_2}{\partial y})h_2 \\
&= \frac{\partial f}{\partial x}h_1 + \frac{\partial f}{\partial y}h_2 \\
&= \frac{\partial f}{\partial x}\frac{h + \bar{h}}{2} + \frac{\partial f}{\partial y}\frac{h - \bar{h}}{2i} \\
&= (\frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y})h + (\frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y})\bar{h} \\
&= \frac{\partial f}{\partial z}h + \frac{\partial f}{\partial \bar{z}}\bar{h} \text{ where } \frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}) \text{ and } \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y})
\end{aligned}$$

Note that if f is holomorphic, then $\frac{\partial f}{\partial \bar{z}} = 0$ and so $f_*(h) = \frac{\partial f}{\partial z}h$. Thus, orientation-preserving conformal mappings in the hyperbolic model are holomorphic mappings. Similarly, orientaion-reversing conformal mappings are antiholomorphic mappings.

A special type of holomorphic function is a Möbius transformation.

Definition A *Möbius transformation* is a holomorphic function $f(z)$ of the form $f(z) = \dfrac{az+b}{cz+d}$ and $ad - bc \neq 0$.

We note that: $f'(z) = \dfrac{ad-bc}{(cz+d)^2}$ and $f^{-1}(z) = \dfrac{-dw+b}{wc-a}$.

A fundamental theorem from complex analysis [2] is:

Theorem. *If $f\colon \mathbf{H}^2 \to \mathbf{H}^2$ is holomorphic, 1-1, and onto, then f is a Möbius transformation.*

Every Möbius transformation is uniquely determined by a 2-by-2 invertible ($ad - bc \neq 0$) complex matrix. The Möbius transformation $f(z) = \dfrac{az+b}{cz+d}$ corresponds to the complex matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We show that the composition of two Möbius transformations can be expressed as the corresponding matrix product of complex matrices.

Let $f(z) = \dfrac{(az+b)}{(cz+d)}$ and $g(w) = \dfrac{(Aw+B)}{(Cz+D)}$ be Möbius transformations. Then

$$\begin{aligned} g \circ f = g(f(z)) &= \frac{Af(z)+B}{Cf(z)+D} \\ &= \frac{A((az+b)/(cz+d))+B}{C((az+b)/(cz+d))+D} \\ &= \frac{A(az+b)+B(cz+d)}{C(az+b)+D(cz+d)} \\ &= \frac{(Aa+Bc)z+(Ab+Bd)}{(Ca+Dc)z+(Cb+Dd)}. \end{aligned}$$

Notice that $\dfrac{(Aa+Bc)z+(Ab+Bd)}{(Ca+Dc)z+(Cb+Dd)}$ corresponds with matrix multiplication

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (Aa+Bc) & (Ab+Bd) \\ (Ca+Dc) & (Cb+Db) \end{bmatrix}.$$

There are three types of Mobius transformations. They are translations, multiplications, and divisions. They and their associated matrices are described as such:

Translation:

$$f(z) = z+b \quad b \in \mathbf{R} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Multiplication:

$$f(z) = az \quad a > 0 \in \mathbf{R} \qquad \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

Division:

$$f(z) = -1/z = \bar{z}/|z|^2 = -x + iy/|z|^2 \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note that each of the matrices associated to each example has det>0 and all entries are real.

Theorem. *Every Möbius transformation can be represented as a composition of the three types of transformations.*

Proof. Case 1: If $c = 0$, then $f(z) = az + b$. Thus, $f(z)$ is a translation of a multiplication.
Case 2: If $c \neq 0$, then $f(z) = \frac{a}{c} + \frac{b + (a/c)d}{cz + d} = \frac{a}{c} + \frac{ad - bc}{c^2}\frac{-1}{z - d/c}$. Thus, $f(z)$ is translation of z by $-d/c$, followed by division by $\frac{-1}{z}$, followed by multiplication by $\frac{(ad - bc)}{c^2}$, followed by translation by $\frac{a}{c}$. ∎

An important class of Möbius transformations are those which preserve the upper half plane. Such a transformation must map the real axis onto itself. That is, $(ax + b)/(cx + d)$ has to be real for all real x. This implies that we may assume a,b,c,d are all real. Now compute the imaginary part of $f(z)$: $\mathrm{Im} f(z) = \frac{ad\,\mathrm{Im}z - bc\,\mathrm{Im}z}{|cz + d|^2} = \frac{(ad - bc)\,\mathrm{Im}z}{|cz + d|^2}$. So we must have $ad - bc > 0$ in order that $f\colon \mathbf{H}^2 \to \mathbf{H}^2$.

Summary: The Möbius transformations which preserve the upper half plane ($\mathrm{Im}z > 0$) can be taken to be of the form $f(z) = (az + b)/(cz + d)$ with a,b,c,$d \in \mathbf{R}$, and $ad - bc > 0$. Moreover, $\mathrm{Im} f(z) = \frac{(ad - bc)\mathrm{Im}z}{|cz + d|^2}$.

Theorem. *Mobius transformations that map $\mathbf{H}^2$ to itself under the composition function operation are isomorphic as a group to the group of matrices $GL_2(\mathbf{R})^+ = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a, b, c, d \in \mathbf{R},\right.$ $\left.\det > 0\right\}$ under matrix multiplication.*

An interesting property of $\mathbf{H}^2$ is that any point in it can be mapped to any other point in $\mathbf{H}^2$ by one of these special Möbius transformations. It follows that any hyperbolic line can be mapped to any other hyperbolic line by means of a special Möbius transformation. The next example shows this by mapping a semicircle onto the positive imaginary axis.

Example. Given two points $z_1,\ z_2 \in \mathbf{H}^2$ we can find the Möbius transformation which maps $z_1 \mapsto i$ and $z_2 \mapsto \alpha i$ for all $\alpha > 0$. First, construct the circle through z_1 and z_2 which is orthogonal to the real axis. (This must map to a vertical straight line to preserve orthogonality to the real axis.) Then $f(z) = \dfrac{z - w_1}{zw_2}$ and $\det f(z) = (w_1 - w_2) > 0$. Thus f is the correct type of Möbius transformation and it maps the semicircle onto the positive imaginary axis.

Now, let $\beta f(z_1) = i$. So, $g(z) = \beta f(z)$ produces $g(z_1) = i$. If $g(z_2)$ lies above i then we are finished. If not, divide by $-1/g(z)$. This will preserve i, and so we are done.

Theorem. *Every Möbius transformation maps circles to circles.*

(Convention: A circle which contains infinity is a straight line.)

Proof. The translation case is trivial. To prove the multiplication case, let a circle in complex notation be denoted by $z = z_0 + re^{i\theta}$. Applying a multiplication to z gives $az = az_0 + are^{i\theta}$ which is a circle with center az_0 and radius $|a|r$. Finally, to show the translation case, let $|z|^2 - (z_0|\bar{z} + \bar{z_0}z) = r^2 - |z_0|^2$ represent a circle in complex notation. Applying $w = f(z) = -1/z$ to z gives

$$r^2 - |z_0|^2 = |-1/w|^2 + (z_0/w + \bar{z_0}/w)$$

$$(r^2 - |z_0|^2)|w|^2 = 1 + (z_0 w + \bar{z}\bar{w}).$$

Case 1: $r^2 - |z_0|^2 = 0$ gives $1 + (z_0 w + \bar{z}\bar{w}) = 0$, a straight line.

Case 2: $r^2 - |z_0|^2 \neq 0$ gives $|w|^2 - w_0\bar{w} + \bar{w}_0 w) = -1/r^2 - |z_0|^2$ where $w_0 = \bar{z}_0/r^2 - |z_0|^2$, a circle.∎

Interestingly, Möbius transformations preserve length in the hyperbolic plane. In fact, Möbius transformations turn out to be the orientation-preserving isometries of the hyperbolic plane. In order to show this we must first understand what is meant by the term length in the hyperbolic plane.

Differential Approach

Length

Recall that the Euclidean length of a parameterized curve γ is defined by

$$L(\gamma) = \int_a^b \sqrt{\gamma'(t)\cdot\gamma'(t)}\,dt = \int_a^b \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2}\,dt$$

where $\gamma\colon [a,b] \mapsto \mathbf{R}^2$ and $\gamma(t) = \gamma_1(t) + \gamma_2(t)$. Also recall that $f'(z) = ad - bc/(cz+d)^2$ are the Möbius transformations which preserve the upper half plane. Furthermore, $f'(z)h = f_*$ is the linear mapping from $T_z(\mathbf{C}) \to T_{f(z)}(\mathbf{C})$. We have shown this mapping is conformal but it does not preserve Euclidean lengths. Thus, we need to develop the idea of a metric in order to redefine lengths of tangent vectors so that f_* does indeed preserve Euclidean lengths. A *metric* is a differentiable function giving the infinitesimal distance ds between any two points.

Recall that the Euclidean length $|f_*(h)| = |f'(z)||h| = \dfrac{ad-bc}{|cz+d|^2}|h|$. We develop hyperbolic length by modifying $|h|$ so that the modified $|f_*(h)| =$ modified $|h|$. Recall $\mathrm{Im}\dfrac{(az+b)}{(cz+d)} = \dfrac{(ad-bc)}{|cz+d|^2}\mathrm{Im}z =$ $|f'(z)|\mathrm{Im}z$ Hence we divide $|f_*(h)|$ by $\mathrm{Im}f(z)$. This gives $\dfrac{|f_*(h)|}{\mathrm{Im}f(z)} = \dfrac{|h|}{\mathrm{Im}z}$. So we define the hyperbolic length of a tangent vector h at z to be $||h|| = \dfrac{|h|}{\mathrm{Im}z}$, i.e., $||f_*(h)|| = ||h||$.

Thus, the infinitesimal hyperbolic distance $ds = |dz|/y = \frac{\sqrt{dx^2+dy^2}}{y}$.

From this, we can compute the *hyperbolic length* L of $\gamma\colon [a,b] \mapsto \mathbf{H}^2$ where $\gamma = \gamma_1(t) + \gamma_2(t)$:

$$L = \int_a^b \frac{\sqrt{d\gamma_1^2 + d\gamma_2^2}}{\gamma_2(t)}\,dt.$$

Remark. (1.) Note that the infinitesimal distance in $\mathbf{H}^2$, $ds = \frac{\sqrt{dx^2+dy^2}}{y}$, is simply the Euclidean distance divided by y. Hence, the ratio between the two is a constant, independent of direction. Thus, angle, which is determined by the side lengths of infinitesimal triangles, is the same when determined by either distance function.

(2.) This also highlights the fact that hyperbolic arc length is the same as Euclidean arc length divided by y. So, in a sense hyperbolic arc length is "longer" than Euclidean arc length.

Having a metric enables us to calculate the lengths of curves. The following examples illustrate this point.

Example. Calculate the hyperbolic arc length between the point i and $1+2i$. We can express this line as $y = x + 1$, $0 < x < 1$ and also as $z = x + iy = x + i(x+1)$. Hence, $dz = dx + idx$ and $|dz| = |(1+i)dx| = \sqrt{2}dx$. So, $||dz|| = |dz|/y = \sqrt{2}dx/(x+1)$. Integrating gives

$$L = \int_0^1 \sqrt{2}dx/(x+1) = \sqrt{2}\ln(x+1)|_0^1 = \sqrt{2}\ln 2.$$

Theorem. *The infinitesimal distance ds^2 is preserved under a Möbius transformation.*

Proof. Let $T = \dfrac{az+b}{cz+d}$. We let $T_*(ds^2)$ represent T applied to the infinitesimal distance.

$$\text{Then,}\quad T_*(ds^2) = \frac{|T'(z)|^2}{\mathrm{Im}(T(z))^2} = \frac{(|T'(z)|dz)^2}{(|T'(z)|\mathrm{Im}z)^2} = \frac{dz^2}{(\mathrm{Im}z)^2}.$$

Theorem. *The hyperbolic arc length of curves is invariant under Möbius transformations. Thus, the Möbius transformations are isometries of the hyperbolic plane.*

As in Euclidean geometry geodesics are defined to be the curves of shortest length. In order to see what these curves look like in the hyperbolic plane we must observe which curves are of shortest hyperbolic arc length. There are two types of geodesics in $\mathbf{H}^2$; vertical straight lines that intersect the imaginary axis and semicircles that are centered on the $x-$axis.

Theorem. *Hyperbolic geodesics are vertical line segments or reparameterizations of vertical line segments.*

Proof. Recall the proof for geodesics in Euclidean geometry. We adjust the proof using hyperbolic arc length. Note that we have already shown that arcs of circles centered on the x-axis can be transformed into a vertical line segment by a Möbius transformation. Thus, to show that arcs of circles centered on the x-axis are geodesics, it suffices to prove that vertical line segments, are hyperbolic geodesics.

Examine the vertical straight line, say the line between the two points ai and bi, for $0 < a < b$. It is an easy exercise to show that the length of the line segment is $\ln(b/a)$.

Now consider any curve in the upper half plane starting at ai and ending at bi. Assume it is parameterized by $a \le t \le b$ with $x = x(t)\, y = y(t)$. Then

$$\begin{aligned}
L &= \int_a^b |dz|/y \\
&= \int_a^b |x'(t)dt + iy'(t)|/y(t)dt \\
&= \int_a^b \sqrt{x'(t)^2 + y'(t)^2}/y(t)dt \\
&\ge \int_a^b \sqrt{y'(t)^2}/y(t)dt \\
&= \int_a^b |y'(t)|dt/y(t)dt \\
&\ge \int_a^b y'(t)dt/y(t) \\
&= \ln y(t)|_a^b \\
&= \ln b/a.
\end{aligned}$$

Thus, the vertical line segments give the shortest length, $\ln b/a$. If $L = \ln b/a$, then $x'(t) = 0$ and $y'(t) \ge 0$. Hence, the only shortest curve from ai to bi is a reparameterization of a vertical line segment.

Thus we know all of the geodesics.

Introduce the notation $d(z_1, z_2)$ = hyperbolic distance between z_1 and z_2 Thus, we know $d(ai, bi) = |\ln b/a|$. This function is a differentiable function giving the distance between two points. It satisfies the following properties:

$$d(z_1, z_2) = d(z_2, z_1)$$

$$d(z_1, z_2) > 0 \text{ if and only if } z_1 \ne z_2$$

$$d(z_1, z_2) \le d(z_1, z_3) + d(z_3, z_2).$$

Hence $d(z_1, z_2)$ is a metric for $\mathbf{H}^2$

This idea of a metric allows us to define a method of calculating hyperbolic lengths in terms of Euclidean distance. The method we will use will be to rewrite $\ln(b/a)$ to obtain $e^{d(z_1, z_2)} = b/a$

Note: $|z_1 - z_2| = b - a$ and $|z_1 - \bar{z}_2| = b + a$. Hence,

$$e^{d(z_1, z_2)} = \frac{|z_1 - \bar{z}_2| + |z_1 - z_2|}{|z_1 - \bar{z}_2| - |z_1 - z_2|}.$$

Example. Let $z_1 = i \quad z_2 = 1 + 2i$ Then

$$|z_1 - \bar{z}_2| = |i - 1 + 2i| = \sqrt{10}$$

$$|z_1 - z_2| = |i - 1 + 2i| = \sqrt{2}$$

Hence,

$$e^{d(z_1, z_2)} = (3 + \sqrt{5})/2$$

Thus, $d(z_1, z_2) = \ln{(3 + \sqrt{5})/2}$

Thus we have developed the idea of finding hyperbolic lengths in terms of Euclidean distance. In general, the formula is valid if $z_1 = ai$ and $z_2 = bi, \quad b > a$. If not, note that both sides of the formula remain unchanged if we apply Möbius transformations. So, if it is not true that $z_1 = ai$ and $z_2 = bi, \quad b > a$ then apply a Möbius transformation to make $z_1 = ai$ and $z_2 = bi, \quad b > a$.

Having a method to calculate lengths enables us to develop some formulas in hyperbolic trigonometry. We define the following:

$$\sinh x = (e^x - e^x)/2$$

$$\cosh x = (e^x + e^{-x})/2$$

$$\tanh x = (e^{2x} - 1)/(e^{2x} + 1)$$

By substitution we obtain

$$\sinh(d/2) = |z_1 - z_2|/2\sqrt{\operatorname{Im} z_1 \operatorname{Im} z_2}$$

$$\cosh(d/2) = |z_1 - \bar{z}_2|/2\sqrt{\operatorname{Im} z_1 \operatorname{Im} z_2}$$

$$\tanh(d/2) = |z_1 - z_2|/|z_1 - \bar{z}_2|$$

Claim. A hyperbolic disc of center z_0 and radius r can be represented by a Euclidean disc with center= $a + b\cosh r$ and radius= $b\sinh r$.

Proof. Let $z_0 = a + ib$ $z = x + iy$. Then $d(z_0, z) = r$.

$$\text{So } \cosh^2(r/2) = \cosh^2\left(\frac{d(z_0,z)}{2}\right) = \left(\frac{|z-\bar{z}_0|}{2\sqrt{Imz\,Imz_0}}\right)^2$$

$$= \frac{(x-a)^2 + (y-b)^2}{4yb}$$

$$\text{Then, since } \cosh^2(r/2) = \frac{\cosh(r) - 1}{2}$$

$$\text{we obtain } (x-a)^2 + (y-b)^2 = \frac{(\cosh(r) - 1)}{2} 2yb$$

$$= (x-a)^2 + (y^2 - b(2\cosh(r))y + b^2 = 0$$

$$= (x-a)^2 + (y - b(2\cosh(r)))^2 - b^2\cosh^2(r).$$

Hence, a hyperbolic circle is a Euclidean disc with center=$(a, b\cosh(r))$ and radius=$b\sinh(r)$.∎

Hyperbolic Area

We can find the area of a hyperbolic region using a similar method to finding area in Euclidean geometry. Recall the double integral formula for Euclidean area:

$$A(D) = \iint_D dA = \iint_D dx\,dy,$$

where dx represents the Euclidean width of the region and dy represents the Euclidean height of the region. The area of a region in the $\mathbf{H}^2$ model is found using the hyperbolic width dx/y and the hyperbolic height dy/y. Hence, the double integral formula for finding the $\mathbf{H}^2$-area of a region D is:

$$A(D) = \iint_D dA = \iint_D dx\,dy/y^2.$$

Example. Assume we have a triangle with all three vertices at infinity.

$$\iint dxdy/y^2 = \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\infty} dy/y^2\,dx$$

$$= \int_{-1}^{1} -1/y\Big|_{\sqrt{1-x^2}}^{\infty} dx$$
$$= \int_{-1}^{1} 1/\sqrt{1-x^2}\, dx$$
$$= \int_{\pi}^{0} \sin\theta/\sin\theta\, d\theta$$
$$= \pi.$$

Theorem. *Möbius transformations that preserve length will preserve area as well.*

Proof. We mimic the proof of $|dz|/\mathrm{Im}z$ as an invariant to show $dA = \mathrm{Im}\,(dzd\bar{z})/2i(\mathrm{Im}\,z)^2$ is invariant. Recall that Euclidean area is invariant under Möbius transformations. This is because

$$A(w(D)) = \iint_{w(D)} dxdy = \iint_{D} |det(Dw)|dxdy.$$

Note $dxdy = 1/2i(dx+idy)(dx-idy) = -1/2idz\,d\bar{z}$.

$$\text{Thus, } A(D) = \iint_D 1/2i\frac{dzd\bar{z}}{(\mathrm{Im}z)^2}.$$

We want to find $A(w(D))$, so we let $w = az+b/cz+d$. This makes $dw = \dfrac{-dz}{|cz+d|^2}$ and $d\bar{w} = \dfrac{dz}{|c\bar{z}+d|^2}$. Since $c, d \in \mathbf{R}$, $|cz+d|^2 = |c\bar{z}+d|^2$. Hence, $dw\, d\bar{w} = \dfrac{dz\, d\bar{z}}{|cz+d|^4}$. But

$$\frac{dw\, d\bar{w}}{|\mathrm{Im}w|} = \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}$$
$$= \frac{(ac|z|^2+bd)+(bc\bar{z}+adz)}{|cz+d|^2}.$$

Now note $bc\bar{z}+adz = (bc+ad)\bar{z}+i(ad-bc)z$ and observe that the imaginary part of this is simply a Möbius transformation that has det $= 1$. Hence, $\mathrm{Im}w = \dfrac{\mathrm{Im}z}{|cz+d|^2}$. Hence, $\dfrac{dw\, d\bar{w}}{|\mathrm{Im}w|} = \left(\dfrac{dz\, d\bar{z}}{|cz+d|^4}\Big/\dfrac{\mathrm{Im}z}{|cz+d|^2}\right) = \dfrac{dz\, d\bar{z}}{(\mathrm{Im}z)^2}$. Thus, $A(D) = A(w(D))$.∎

Isometric Approach

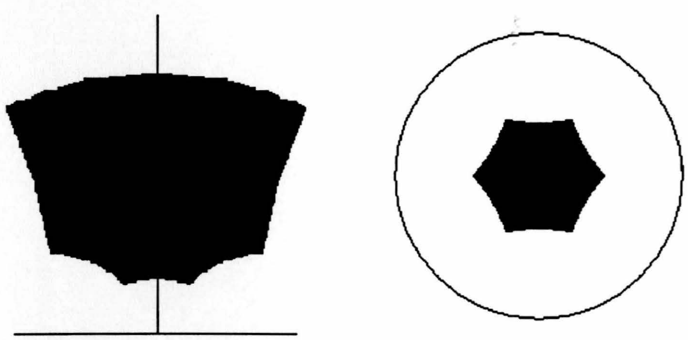
Recall the three types of Möbius transformations: translations, multiplications, and divisions. Having a metric allows us to see that each of these is a hyperbolic isometry. Recalling our Euclidean isometries, it seems that we have forgotten hyperbolic rotations. These are hard to understand in the $\mathbf{H}^2$ model. For this reason, we introduce a second model space for hyperbolic geometry, the unit disc model space, denoted $\mathbf{D^2}$.

Definition. A *biholomorphism* is a 1-1, onto, holomorphic map with holomorphic inverse.

Theorem. *The function* $f(z) = \dfrac{iz+1}{z+i}$ *a biholomorphism that maps* $\mathbf{H}^2$ *to the unit disc.*

Proof. The Möbius transformation $w = J(z) = (iz+1)/(z+i)$ maps the upper half plane to the unit disc with all points on the real axis mapping to the boundary of the unit disc. It is obviously 1-1 and onto. w is holomorphic because $J'(z) = \dfrac{\partial f}{\partial z}$ and J^{-1} is also easily shown to be holomorphic. ∎

Remark. An important characteristic of the unit disc model is that its natural boundary, the unit circle, represents the circle at infinity. The points that are on this circle are not really points of $\mathbf{D^2}$ but rather limits of points in $\mathbf{D^2}$. The translation of this line in $\mathbf{H}^2$ corresponds to the imaginary axis.

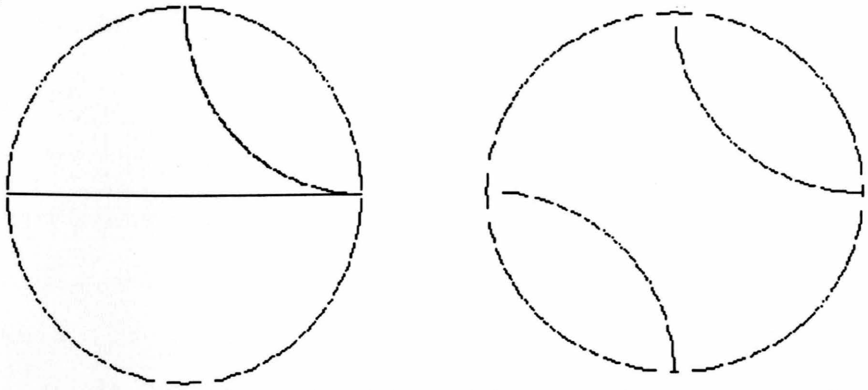


Applying $J(z) = (iz+1)/(z+i)$ to a hexagon in $\mathbf{H}^2$

The unit disc model is useful in exhibiting the fact that there are two types of parallel lines in hyperbolic geometry due to the hyperbolic parallel postulate.

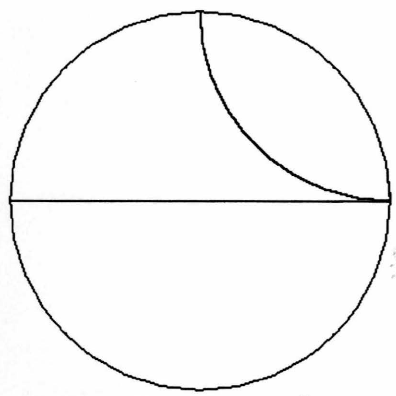
Definition: Two geodesics which do not intersect are *asymptotically* parallel if they share a point at infinity.

Definition: Two geodesics which do not intersect are *disjoint* parallel if they share no points at all.



Asymptotically (l) and Disjoint (r) Parallel Lines in the $\mathbf{D}^2$-model

We define $\mathbf{D}^2$-distance between points $w_1, w_2 \in \mathbf{D}^2$ to be the $\mathbf{H}^2$ distance between $J^{-1}(w_1), J^{-1}(w_2) \in \mathbf{H}^2$. $\mathbf{D}^2$-lines are circular arcs orthogonal to the unit circle bounding $\mathbf{D}^2$. This description includes the diameters of $\mathbf{D}^2$ (which are Euclidean line segments.)



Two Types of $\mathbf{D}^2$ Lines

The isometries of $\mathbf{D}^2$, then, are the conjugates JhJ^{-1} of $\mathbf{H}^2$-isometries h. To show rotations are isometries, where rotations about the origin in $\mathbf{D}^2$ are defined by $r_\theta(w) = e^{i\theta}w$ for $\theta \in \mathbf{R}$, we calculate the $\mathbf{D}^2$ distance.

Since $w = J(z) = (iz+1)/(z+i)$ we have $z = (-w+1)/(w-i)$. Hence,

$$\begin{aligned}
\frac{|dz|}{\mathrm{Im} z} &= \frac{|d(-iw+1)|}{(w-i)} \Big/ \mathrm{Im}\frac{(-iw+1)}{(w-i)} \\
&= \frac{|-2dw|}{(w-i)^2} \Big/ \mathrm{Im}\frac{(1-iw)(\bar{w}-\bar{i})}{(w-i)(\bar{w}-\bar{i})} \\
&= \frac{|2dw|}{|w-i|^2} \Big/ \mathrm{Im}\frac{(1-iw)(\bar{w}+i)}{|w-i|^2} \\
&= \frac{|2dw|}{(1-|w|^2)}.
\end{aligned}$$

It is easy to show that $\mathbf{D^2}$-distance remains invariant under Euclidean rotations because $|w|$ remains unchanged.

More generally, $\frac{|2dw|}{(1-|w|^2)}$ remains invariant under Euclidean reflections in a line through the origin. Because of these invariants, we define the $\mathbf{D^2}$ reflections (in the $x-$axis) to be $\bar{r}(w) = \bar{w}$.

As in Euclidean geometry, reflections in hyperbolic geometry are extremely important. Since reflection in $\mathbf{D^2}$ about the real axis is $\bar{r}(w) = \bar{w}$, reflection in $\mathbf{H^2}$ is its conjugate by J^{-1}. Doing this calculation gives an $\mathbf{H^2}$ reflection about the unit circle to be $z = 1/\bar{z}$. It is possible to reflect about any circle centered at (α, ρ). We can do this using $\mathbf{H^2}$-isometries, $t_\alpha d_\rho I d_\rho^{-1} t_\alpha^{-1}$. These reflections, together with the Euclidean reflections $t_\alpha \bar{r}_{OY} t_\alpha^{-1}$ make up all of the $\mathbf{H^2}$ reflections. We will discuss further the importance of reflections after a brief look at the geometric description of some hyperbolic isometries.

Geometric Description of Isometries

It is very useful to understand the geometric meaning of each hyperbolic isometry. Stillwell [7, pg. 80] states, "Every $\mathbf{H^2}$ isometry, when conjugated to a suitable position, becomes a Euclidean mapping. This is very helpful in visualizing hyperbolic geometry." We note that by describing the isometries geometrically we are able to observe that each of the two models has its advantages for visualizing different isometries. Some are better understood in the $\mathbf{H^2}$ model and some are better understood in the unit disc model.

Rotation about the origin in the unit disc model space is $r_\theta(w) = e^{i\theta}w$. Its effect is easily seen in the unit disc model. Rotation permutes the diameters of lines through the origin and leaves invariant the circles centered at the origin. The effect in the $\mathbf{H^2}$ model is the image of the unit disc rotation and is more complicated. It suffices to say that a $\mathbf{H^2}$- rotation is not a Euclidean rotation in the sense of rigid (Euclidean) rotation about one point. Rather, circles get mapped to circles and lines get mapped to lines but in a more complicated manner.

The Möbius transformation $t_a(z) = a + z$ denotes a *limit rotation* about infinity in $\mathbf{H^2}$. A limit rotation is easy to understand in the $\mathbf{H^2}$ model. Note that analytically t is exactly the transformation which represents a Euclidean translation. Its effect in $\mathbf{H}^2$ is to permute the lines x=constant and leave invariant the y =constant lines (known as horocycles.) Horocycles have no analogues in Euclidean geometry. Limit rotations leave no fixed points in either of the model spaces but there is a fixed point at infinity which is the common end of the permuted lines.

The Möbius transformation $f(z) = az\,, a > 0$ represents *translation* in $\mathbf{H}^2$ along the imaginary axis. Note that this isometry is analytically the transformation which represents a Euclidean multiplication. A hyperbolic translation permutes the semicircles centered at the origin and leaves invariant the y-axis and the lines y =(constant)$\cdot(x)$. There is no fixed point in either model space but there are two on the circle at infinity at the ends of the invariant ($\mathbf{H}^2$- or $\mathbf{D}^2$-) line.

The Möbius transformation that represents division is the simplest example of a *glide reflection* in $\mathbf{H}^2$, a trivial reflection. The simplest case of a glide reflection is a translation with a reflection in the y-axis. A glide reflection, more generally, is the product of a reflection with translation whose axis is the line of reflection. It is described as $f(\bar{z}) = a + \frac{a^2}{|a|^2}\bar{z}$, where $a = r_0 e^{i\theta_0}$. The description of a translation in $\mathbf{H}^2$ holds as a description of a glide reflection in $\mathbf{H}^2$ as well. Similarly, a glide reflection has two fixed points at infinity.

This dependence between hyperbolic model spaces is obviously useful in understanding hyperbolic geometry. As another example, we can now define $\mathbf{H}^2$-rotations and $\mathbf{H}^2$-reflections by conjugating the $\mathbf{D}^2$-rotations and reflections by J. $\mathbf{H}^2$-rotations are about the point i and $\mathbf{H}^2$-reflections are in the unit circle. Reflections are important to us for many reasons. Most importantly, they give the $\mathbf{H}^2$-lines as the fixed point sets of $\mathbf{H}^2$-reflections. These are Euclidean semicircles in $\mathbf{H}^2$ with centers on the x-axis and Euclidean half lines in $\mathbf{H}^2$.

Lemma. *The set of points $\mathbf{H}^2$-equidistant from two points, $P, P' \in \mathbf{H}^2$ is an $\mathbf{H}^2$-line L, and $\mathbf{H}^2$-reflection in L exchanges P and P'.*

Proof. We choose P, P' to be mirror images in the y-axis. (If not, use rotation to force the positioning.) Hence, reflection $\bar{r}$ in the y-axis exchanges P and P'. Since $\bar{r}$ is an $\mathbf{H}^2$-isometry which fixes each point Q on the y-axis, it follows that any such Q is $\mathbf{H}^2$-equidistant from P, P'.

Hence, we have $\mathbf{H}^2$-length $(P'R') = \mathbf{H}^2$-length (PR) by refelction. Then

$$\begin{aligned}\mathbf{H}^2 - \text{length } (PR) &= \mathbf{H}^2 - \text{length } (P'R) \text{ by hypothesis}\\ &= \mathbf{H}^2 - \text{length } (P'Q) + \mathbf{H}^2 - \text{length } (QR)\\ &= \mathbf{H}^2 - \text{length } (P'Q) + \mathbf{H}^2 - \text{length } (QR') \text{ by reflection.}\end{aligned}$$

This contradicts the triangle inequality imposed by the hyperbolic metric. Hence, the contradiction shows that the y-axis is the complete $\mathbf{H}^2$-equidistant set of P, P'. ∎

It is possible to show that each of the fundamental $\mathbf{H}^2$−isometries are products of $\mathbf{H}^2$-reflections. For example, an $\mathbf{H}^2$ translation, $d_\rho = (1/\bar{z}_{R_1})(1/\bar{z}_{R_2})$, whenever $R_2^2/R_1^2 = \rho$, where R_1, R_2 are the radii of two semicircles in $\mathbf{H}^2$. An $\mathbf{H}^2$-limit rotation is analytically the same as a Euclidean translation. We have already shown that a Euclidean translation is the product of Euclidean reflections. Hence, an $\mathbf{H}^2$-limit rotation is the product of $\mathbf{H}^2$-reflections. From this it follows that a hyperbolic glide reflection is the product of reflections. This is because we defined a hyperbolic glide reflection to be the product of a reflection with a translation. Rotation remains to be shown. It is easiest to see that a rotation is the product of reflections using the unit disc model. Recall that the diameter of the unit circle is a $\mathbf{D}^2$-line. It is also a Euclidean line. Hence, a Euclidean reflection around the diameter of the unit circle is the same as a hyperbolic reflection. Hence, rotation is the product of hyperbolic reflections. From this we conjecture the following ([7], p.88)

Theorem. *Each $\mathbf{H}^2$ isometry is the product of one, two, or three $\mathbf{H}^2$ reflections.*

Corollary. *The $\mathbf{H}^2$ isometries form a group.*

Proof. This is similar to the Euclidean case because each reflection is a self inverse.

This description of isometries in terms of reflections recalls the notion of orientation (where orientation-preserving and orientation-reversing isometries form complementary sets.) In Euclidean geometry, an isometry is orientation-preserving if it is the product of two reflections and orientation-reversing if it is the product of one or three reflections. This is similar in hyperbolic geometry.

Theorem (Poincaré). *([7], pg. 90) The* $\mathbf{H}^2$*-isometries are of the form*

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbf{R}$ *and* $ad - bc = 1$ *(orientation-preserving) and*

$$\bar{f}(z) = \frac{-a\bar{z}+b}{-c\bar{z}+d},$$

where $a, b, c, d \in \mathbf{R}$ *and* $ad - bc = 1$ *(orientation-reversing).*

Corollary. *The* $\mathbf{D}^2$*-isometries are the functions*

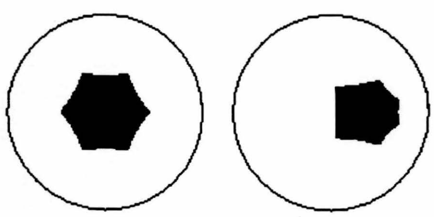
$$f(z) = \frac{az+b}{\bar{b}z+\bar{a}}$$

where $a, b \in \mathbf{C}$ *and* $|a|^2 - |b|^2 = 1$ *(orientation-preserving) and*

$$\bar{f}(z) = \frac{a\bar{z}+b}{\bar{b}\bar{z}+\bar{a}}$$

where $a, b \in \mathbf{C}$ *and* $|a|^2 - |b|^2 = 1$ *(orientation-reversing).*

Transformation of a hexagon in the $\mathbf{H}^2$-model



Transformation of a hexagon in the $\mathbf{D}^2$-model

The only orientation-reversing isometry is a glide reflection because it is the product of either one or three reflections. It is more complicated in hyperbolic geometry to prove this because of the fact that there is more than one type of parallel. Stillwell ([7], p. 96) shows that one must prove orientation-preserving and orientation-reversing for three cases in hyperbolic geometry. The cases are two intersecting lines, two asymptotic lines, and two disjoint parallel lines.

Theorem (Classification of Hyperbolic Isometries). *Each isometry of the hyperbolic plane is either a rotaion, limit rotation, translation, or a glide reflection.*

This is the final result for which we were aiming. By mirroring the development of Euclidean geometry, we used differential calculus to develop a hyperbolic metric. This allowed us to consider the hyperbolic isometries and hence describe hyperbolic geometry. In so doing we developed two isomorphic model spaces for hyperbolic geometry, $\mathbf{H}^2$ and $\mathbf{D}^2$, which aided in visualizing the world of hyperbolic geometry.

APPENDIX

	EUCLIDEAN	HYPERBOLIC
Model Space	$\mathbf{R}^2$	$\mathbf{H}^2$
ds^2	$ds^2 = dx^2 + dy^2$	$ds^2 = \dfrac{dx^2 + dy^2}{y^2}$
Arc Length	$L = \int \sqrt{dx^2 + dy^2}$	$L = \displaystyle\int \frac{\sqrt{dx^2 + dy^2}}{y}$
Area of Triangle	$A = \alpha + \beta + \gamma$	$A = \pi - (\alpha + \beta + \gamma)$
Area of Disk	$A = \pi(radius)^2$	$A = 4\pi \sinh(radius/2)^2$
Fundamental Isometries		
Translations	$t_\alpha(z) = \alpha + z$	$d_\rho = \rho z$
Rotations	$r = e^{i\theta} z \quad (-\pi < \theta \leq \pi)$	$r(z) = \dfrac{\cos(\theta/2)z - \sin(\theta/2)}{\sin(\theta/2)z + \cos(\theta/2)}$
Glide Reflections	$\bar{r}(z) = e^{i\theta}\bar{z} + z_0$	$\bar{r}(z) = r_0 e^{i\theta_0} + \dfrac{r_0 {e^{i\theta_0}}^2}{\lvert r_0 e^{i\theta_0}\rvert^2}\bar{z}$
Limit Rotations (about ∞)	None	$t_\alpha(z) = \alpha + z$
Trigonometry Laws		
Law of Cosines	$c^2 = a^2 + b^2 - 2ab\cos(\alpha)$	$\cosh(c) = \cosh(a)\cosh(b) - \sinh a \sinh b \cos(\alpha)$
		$\cos(\gamma) = -\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\cosh c$
Law of Sines	$\dfrac{\sin(\alpha)}{a} = \dfrac{\sin(\beta)}{b} = \dfrac{\sin(\gamma)}{c}$	$\dfrac{\sin(\alpha)}{\sinh(a)} = \dfrac{\sin(\beta)}{\sinh(b)} = \dfrac{\sin(\gamma)}{\sinh(c)}$
Pythagorean Theorem	$c^2 = a^2 + b^2$	$\cosh c = \cosh a \cosh b$
Identities	$\cos^2(x) + \sin^2(x) = 1$	$\cosh^2(x) - \sinh^2(x) = 1$
	$1 + \tan^2(x) = \sec^2(x)$	$1 - \tanh^2(x) = \operatorname{sech}^2(x)$
	$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$	$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \sinh(x)\sinh(y)$
	$\cos(x \pm y) = \cos(x)\cos(y) \pm \cos(x)\sin(y)$	$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$
	$\sin^2(x/2) = \dfrac{1 - \cos(x)}{2}$	$\sinh^2(x/2) = \dfrac{\cosh(x) - 1}{2}$
	$\cos^2(x/2) = \dfrac{1 + \cos(x)}{2}$	$\cosh^2(x/2) = \dfrac{\cosh(x) + 1}{2}$

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