Statistical Physics and Network Theory as an Interdisciplinary Approach to Legislative Forecasting

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Abstract

The current field of American politics, particularly within the legislative branch, is considered highly polarized and significantly inscrutable. In the face of this shifting political landscape, attempts to rigorously analyze the United States Congress under the standard paradigms of qualitative political science have proved partially insufficient, particularly efforts to accurately forecast legislative behavior. We introduce two classes of models to analyze the voting behavior and political topology of the United States Congress. The first are multi-temperature kinetic Ising models and the second are weighted network models. These models are presented first in order to evaluate and study the partisanship and social interactions within Congress, and second as a means of conveying the versatility of statistical physics and network-based computer simulations in non-physical contexts. The physical and theoretical basis of these models are provided, with special attention paid to the distinct challenges and benefits of applying non-equilibrium statistical physics. The development and legislative forecasting applications of each are then detailed, alongside the results of interest. Each indicate complimentary and contrasting results, which are particularly significant when considered alongside modern conceptions of the American political environment.

Contents

1 Introduction

In the face of government shutdowns, controversial policy decisions, and a decline in the enactment of laws, the United States Legislature has received increased scrutiny throughout the 21st century. Gallup polls conducted as recently as February of 2020 indicate a 23% approval rating of the American legislature [1], while a 2014 report by the Washington Post indicates that these most recent classes of Congress are among the least effective in United States history [2]. Popular criticism continually identifies polarization and lack of cooperation as the primary deficiencies of the legislature. The traditional techniques under which political behavior is studied have only begun to make steps in qualitatively and quantitatively analyzing these phenomena.

Simultaneously, however, government publications, news media outlets, and direct communication between legislators and constituents have provided significant access to data detailing not only the voting records of legislators, but also the external and internal processes which eventually determine those votes. Armed with this wealth of information, physicists have entered the field of political science in demonstrably high numbers [3] [4], pioneering the growing field of 'socio-physics.' This field not only provides a new direction for the development of quantitative political forecasting but also simultaneously capitalizes on this widely available data for the study of complex, cooperative systems. Moreover, this field provides a pedagogical tool to demystify political processes and educate the public on legislative trends. In particular, non-partisan organizations such as GovTrack [5] and PredictGov [6] have gained increased popularity and coverage over the last few years as their sites provide legislative forecasting and analysis of each legislator's voting trends, earning these organizations citations from the *Huffington Post* [7], *New York Times* [8], and even HBO's *Last Week Tonight with John Oliver* [9].

These organizations' simultaneous application of machine learning, network theory, and political science raises the necessary question implicit in our modern technologically and scientifically literate society: how can our disparate stores of scientific and social knowledge be brought together in order to create something greater than the sum of their parts? While this paper cannot claim to represent the answer to that question, it earnestly attempts to mark its place in the endeavor. Voter models have been studied for a number of years in statistical physics, and the application of statistical physics to the modern American political environment may allow for the development of more accurate models. Moreover, as this research stands at the confluence of physics and political science, the versatility of these models promises the possibility of further application in the fields of sociology, epidemiology, legal studies, and other fields which deal in complex networks. Furthermore, each of these topics also intersect with the concept cooperation, understood as behaviors in which individuals act on the behalf of others at a cost to themselves. As such, this thesis also adds to the ongoing effort to explain this conundrum in social science which has been called one of the most significant of our generation, given its apparent contradiction to Darwinian and capitalistic paradigms [10].

The translation of the United States Congress to a numerically soluble statistical system benefits from relatively straightforward analogues to well-known physical and theoretical models. In order to demonstrate this, a brief explanation of the United States Congress's structure and legislative processes is necessary. The United States Congress is a bicameral legislature: it is divided into two discrete legislative bodies, the *House of Representatives* and the *Senate*. The Senate is composed of 100 voting legislators, while the House is comprised of 435 voting legislators. In both the House of Representatives and the Senate, motions to adopt legislation may be determined by a roll call vote, or a vote in which each legislator's vote is individually recorded. Therein, legislators provide votes of 'yea/aye' (approval of the motion) or 'nay' (disapproval of the motion). In the case of a majority vote in the affirmative, the motion is passed. While a supermajority vote may be required under special circumstances, these occur only under certain conditions, and as such they may be considered unnecessary considerations for this brief overview. With this sketch of the legislative process in mind, the United States Congress could be easily conceptualized as a *lattice* or *network* comprised of either 100 or 435 *sites* or *nodes*, respectively. Each site or node is then occupied by a legislator in either the 'yea' or 'nay' states, and therefore the entirety of Congress could be conceptualized as a two state macro-system determined by the majority of states therein.

While this initial framework may be relatively simple, the process by which each legislator resolves to a 'yea' or 'nay' state is a result of internal factors such as the individual's political leaning and agenda, as well as external factors including peer influence, public opinion, and lobbyist groups. These internal factors, though realistically defined on an issue-to-issue basis, can largely be simplified under the umbrella descriptors of liberal and conservative. These terms are defined according to a legislator's general stance on a variety of political issues and how those stances compare to the position of either the liberal Democratic Party or the conservative Republican Party. While this metric for political ideology and agenda discounts the divide between fiscal and social liberalism and conservatism, as well as a variety of complex sociopolitical considerations, it serves as a relatively accurate general descriptor of political leaning.

The external factors which may impact a particular legislator's decision-making are comparably easier to describe without generalization. Interactions between legislators or their respective staffs, between legislators and lobbyists, and between legislators and constituents are well documented and recorded. However, the degree to which a legislator or group of legislators may be susceptible to external influences varies significantly. This trait manifests on a spectrum ranging from reluctance to change stance on a given issue to what is often described as *maverick voting* or *bloc voting* in the most extreme case. In the case of 'maverick' voting, a legislator may refuse to change an initial vote regardless of external influence. Similarly, in the case of 'bloc' voting a group of legislators may elect to vote the same as one another regardless of the legislation in question or external influences. These behaviors are especially noteworthy in that 'maverick' voting is especially emblematic of the intractability frequently commented on in the modern legislature, while 'bloc' voting, which frequently occurs along partisan lines, demonstrates political polarization. On this note, outside of the extreme cases of maverick Representatives or bloc voting groups, external interactions are also significantly limited, though not necessarily eliminated, by party politics and disparities in political ideology. Therefore, legislators of similar political leanings or party affiliations are more likely to influence one another than legislators of extremely disparate political beliefs.

Working within this readily apparent and analogous statistical system within the United States Congress, this thesis presents two classes of cooperative, stochastic social network models in order to describe and analyze voting behavior.

We begin with a presentation on relevant models in section 2, which begins by introducing the concept of a voting model and illustrates its function through a brief overview of an especially simple two-state model. The Ising model is presented in 2.2. The non-equilibrium extensions of the Ising model, kinetic Ising models and multi-temperature kinetic Ising models, are presented in 2.3. Numerical methods significant to solving kinetic Ising models are then provided in 2.4. Next, 2.5 presents an introduction to network theory and network based models. We present an original application of multi-temperature kinetic Ising models in section 3, beginning by detailing the model's development. This is followed by results of the model alongside our interpretation. We then present an original application of network theory in order to model the United States Congress in section 4, detailing development, followed by results and interpretation. Section 5 concludes this thesis with a comparison of results, alongside a brief exploration of further applications for these models and methods.

2 Voter Models

The simplest goal of a voting model is to describe the formation or dissolution of consensus in a population. Before moving on to present our two original classes of voting models, a brief exploration of a well-studied and simple voting model may be beneficial as an introduction to the topic. While a variety of simple voting dynamics may be simulated to this end, we will consider a model which possesses the distinct advantages of being not only solvable in arbitrary spatial dimensions but also topologically versatile enough to demonstrate the application of both spin-flip Ising-type models and network models. This model, first proposed by Holley and Liggett [11], considers a population of voters occupying a system of any given dimensionality. These voters are then graphically arranged according to a stochastic topology, and each is assigned an initial opinion state, q. As the simulation progresses over an arbitrary time scale represented by discrete update events, each voter adopts the opinion state of their nearest neighbor. Throughout each update event over arbitrary time steps, Δt, this model simulates the change in opinion of the voter at lattice site x.

The transition rate of any given voter occupying the position, x, while the system is in configuration s is defined as,

$$w_x(s) = \frac{1}{2}[1 - \frac{s(x)}{z} \sum_{y \in \langle x \rangle} s(y)] \tag{1}$$

where the opinion state of the voter occupying site x is given by $s(x)$. The coefficient z is the number of nearest neighbors to x, known as the lattice coordination number. The set of nearest neighbors to site x is given by $\langle x \rangle$. The sum is therefore taken over all other voters, y, in closest proximity to x. Note that the transition rate of the voter at site x is proportional to the inverse of disagreeing neighbors, where $s(y) \neq s(x)$. Therefore, a voter will only change opinion if a nearest neighbor possesses a different opinion state.

While this transition rate exclusively describes the dynamics of a single voter at each update event, from this rate we are able to derive a master equation describing the transition rates of the entire system. The probability that the set of all voters are in configuration s at time t is provided by the master equation,

$$\frac{d}{dt}p(s) = -\sum_x w_x(s)p(s) + \sum_x w_x(s^x)p(s^x) \tag{2}$$

where s^x denotes the state of the system in which the voter occupying site x has changed opinion. This means that $s^x(x) = s(y)$ when $s(y) \neq s(x)$ or alternatively, $s^x = -s(x)$. Therefore, the gain term relates the probability of all transitions to the configuration s in a single update event, while the loss term relates the probability of all transitions out of configuration s in a single update event.

An example of two possible configurations following an update event, s and s^x, for a system of $N = 3$ voters on a one dimensional lattice, wherein $x = 2$, and opinion states, q, may be either $+1$ or -1 is provided in Figure 1.

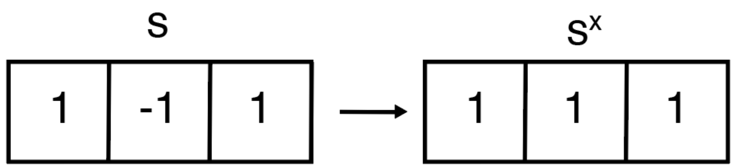

Figure 1. An example of a possible transition between configurations, wherein the opinion state of the voter populating site x is updated.

In order to determine the system's inevitable steady state, we consider the average opinion of the system, S,

$$S = \langle f(s) \rangle = \sum_s f(s)P(s) \tag{3}$$

where $f(s)$ is the set of all voting states.

Drawing from the master equation, we are able to determine the time dependency of S, by considering the average opinion state of any voter at position x over an arbitrary time step, Δt.

$$s(x, t + \Delta t) = \begin{cases} s(x,t) & \text{with probability } 1 - w_x(s)\Delta t \\ -s(x,t) & \text{with probability } w_x(s)\Delta t \end{cases} \tag{4}$$

As the opinion state of the voter at site x changes by $-2s(x)$ at a rate of ω_x, the average opinion evolves according to,

$$\frac{dS(x)}{dt} = -S(x) + \frac{1}{z} \sum_{y \in \langle x \rangle} S(y) \tag{5}$$

Summing this over all sites, we note that the average alignment of the system, $m = \sum_x \frac{S(x)}{N}$ is necessarily conserved. While the alignment of the system throughout each update event does change (as only a single site is updated at a time, the alignment of the configuration necessarily changes), the average over all sites and trajectories of the dynamics are conserved [12]. As a consequence of this conservation, the system must necessarily reach a consensus.

Consider a finite population of voters wherein an initial fraction ρ are in the opinion state $q = +1$, while a fraction $1 - \rho$ are in the opinion state $q = -1$. The initial alignment is therefore $m_0 = 2\rho - 1$. The final alignment is then given by,

$$\begin{aligned} m_{t\to\infty} &= E(\rho) \times 1 + (1 - E(\rho)) \times (-1) \\ &= 2E(\rho) - 1 \end{aligned} \tag{6}$$

where $E(\rho)$ is the exit probability, or the probability of a final consensus on the opinion state, $q = 1$. This expression equals the initial alignment, $m_0 = 2\rho - 1$. According to these conservation dynamics, this voting system must ultimately reach one of two final alignments, with probabilities given by,

$$m = \begin{cases} +1 & \text{with probability } E(\rho) \\ -1 & \text{with probability } 1 - E(\rho) \end{cases} \tag{7}$$

As a result, the probability of reaching a consensus on the opinion state $q = +1$ is $E(\rho) = \rho$ and the probability of reaching a consensus on the opinion state $q = -1$ is $1 - \rho$. As such, the final opinion state depends exclusively on the initial fractions of opinion states, ρ and $1 - \rho$, indicating that the steady state of the model has no dependence on the system size or the topology of the network. The initial and final states of the system given an arbitrary initial alignment, m_0, are shown in Figure 2 and 3 respectively.

While this model is notably simple, and largely unrealistic given voters' immediate willingness to

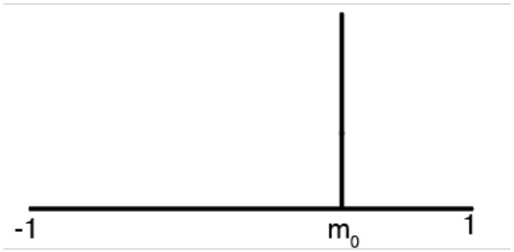

Figure 2. The initial alignment, m_0 of the system, with unit probability.

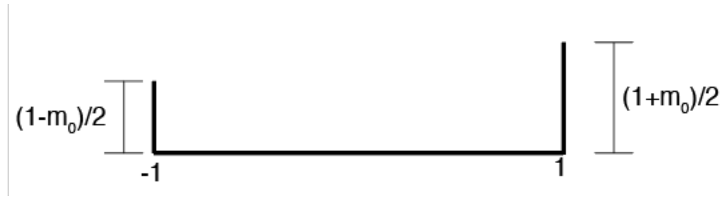

Figure 3. The two possible final alignments of the system ($m = -1$ and $m = 1$), alongside the magnitude of their probability.

adjust their opinion state towards conformity with nearest neighbors, its significance in this thesis is critical. In one-dimension, this model is functionally identical to an Ising-type model. In two-dimensions, this model is a simple application of network theory. In either case, this model succinctly demonstrates the most essential purposes of a voter model. Specifically, that a model must demonstrate the process of forming opinions and identify the dependencies of that process. The following models could therefore be considered adaptations of this simple voting model, in that they attempt to more rigorously define methods of interaction along political lines, and as such generate more nuanced results specific to the United States Congress.

2.1 Ising Models

The first class of models we present are *multi-temperature kinetic Ising models*, a class of non-equilibrium Ising models. While the methods and applications of statistical physics in general are versatile and well-developed, the study of systems out of equilibrium is considerably less mature. Despite considerable effort spread out over more than a century of research, universally applicable analogs to the near-ubiquitous mathematical formalisms standard in equilibrium statistical physics are limited. For example, no analogs of the ubiquitous canonical Boltzmann factor or partition function of equilibrium statistical physics have been found which can be applied with consistency [13]. What strides have been made in the field of non-equilibrium statistical physics are generally concerned with systems at small deviations from equilibrium [12]. That being said, systems out of equilibrium are well appreciated in computer simulation due to their capacity to present conceptually straightforward and explicit results which may be adapted to a variety of systems [15]. As one-dimensional versions of kinetic Ising models can be solved exactly, these models provide a versatile space for simulating soluble complex systems [16].

Given that non-equilibrium systems are considerably more difficult to approach, our treatment of multi-temperature kinetic Ising models will begin with a consideration of the comparably simple equilibrium Ising model. This will not only allow this thesis to better present the distinct advantages and disadvantages of working in non-equilibrium statistical physics, but also provide a significantly more approachable space in which to define significant functions of state and tendencies of Ising models which carry into their non-equilibrium counterparts.

The one-dimensional Ising model, first solved by Ernst Ising in 1925 [17], is an equilibrium model of ferromagnetism. While modern Ising models may be used to simulate both antiferromagnetic and non-magnetic systems [18], we will proceed with a discussion of purely the ferromagnetic model as it most significantly relates to this thesis. Within the Ising model, discrete variables, σ_i, representing magnetic dipole moments of atomic spins are arranged in N discrete, equidistant lattice sites ($i = 1, 2, ..., N$). Critically, the lattice may have either periodic or non-periodic boundaries, often referred to as toroidal or free boundary conditions, respectively [17]. In the case of periodic boundary conditions, the N^{th} spin interacts with the first spin, therefore the one-dimensional lattice is treated as a ring. In the case of non-periodic boundary conditions, the lattice is treated as a chain. For the purposes of this thesis, we will proceed with the assumption that boundary conditions are non-periodic, and refer to the lattice as an Ising chain.

Each spin is in one of two states ($\sigma_i \in \{+1, -1\}$), and each spin is able to interact with its neighbors. Coupling constants, J_{ij}, are introduced to describe these interactions, and are defined such that $0 < J_{ij}$ for ferromagnetic systems. These constants intimate that neighboring spins are able to induce spin transitions in one another, at a rate defined by the system's total change in energy for a given flip. As we have enforced non-periodic boundary conditions, it is worth pointing out here that lattice sites $n = 1$ and $n = N$ will each have one less nearest neighbor given their positions at the beginning and end of the Ising chain, respectively.

The energy of a spin configuration, or any assignment of spin values to each lattice site is given by the Hamiltonian function [17],

$$H(\sigma) = -\sum_{NN} J_{ij}\sigma_i\sigma_j + B\sum_{i=1}^{N}\sigma_i \tag{8}$$

where J_{ij} is the coupling constant between neighboring spins σ_i and σ_j, and the sum is taken over all pairs of adjacent spins. B is an external magnetic field. For the purposes of this thesis, we will proceed with the assumption that the external magnetic field is 0.

The system at equilibrium tends towards the lowest possible energy; therefore, for a ferromagnetic system, the spins tend to align with the orientation of the magnetic field. If there is no external magnetic field, the spin configuration tends towards an equal number of particles aligned in either states [18]. The total magnetization, or the total alignment of spins, is given by the sum of all spin states [17],

$$M = \sum_{i}^{N} \sigma_i \tag{9}$$

Therefore, at equilibrium and absent external a magnetic field, the magnetization tends to 0, while the entropy is maximized.

Considering the Hamiltonian under these assumed conditions, the energy of the non-periodic lattice site chain is given by [18],

$$E = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} \tag{10}$$

in which case the partition function for N particles can be solved as [17],

$$Z_N = (2cosh(\beta J))^{N-1} \tag{11}$$

where β is the inverse temperature $\frac{1}{k_b T}$, in which k_b is Boltzmann's constant. The partition function relates the probability of the Ising chain existing in any configuration, σ, by $P_\sigma = \frac{1}{z} e^{-\beta E_s}$. Given this, the equilibrium properties of the Ising model follow relatively clearly [18] and are provided in Figure 4.

This application of equilibrium statistical physics, while largely inapplicable for systems outside of equilibrium, does provide a critical insight. Specifically, that the Boltzmann weight should provide a solution for non-equilibrium systems as they move towards the steady state [13].

It should also be noted here that, as can be observed from a cursory examination of the Ising

Mean Internal Energy ($\langle E \rangle$)	$-\frac{\partial Z}{\partial \beta}$
Heat Capacity (C_v)	$\frac{\partial \langle E \rangle}{\partial T}$
Entropy (S)	$k_B(lnZ + \beta\langle E \rangle)$
Free Energy (F)	$-k_B T ln(Z)$

Figure 4. The relation between the partition function and functions of state.

Hamiltonian and subsequently derived partition function, increases in the temperature of a lattice yield increases in the system's entropy, indicating that temperature increases the spectrum of possible spin states at any given time.

Furthermore, the partition function provides a means by which to define the relationship between spin states. The relation between two spins is described by the spin-spin correlation function, $\langle \sigma_i \sigma_{i+r} \rangle$, which notes the likelihood of a spin i and another spin r lattice sites away to point in the same direction or in opposite directions at equilibrium.

This value, taken alongside the magnetization of the Ising chain, is perhaps the most immediately significant to this thesis. As we intend to treat an Ising chain as analogous to the United States legislature, the interdependence of the constituent particles' spin states is a vital concept to understanding how an Ising-type model might serve as an analog to the influence between legislators' voting states.

The most critical insight here is the Ising model's assumption of exclusively local interactions between particles [12]. With this in mind, we may reasonably assume that spins located in closer proximity within the lattice possess a larger correlation between spin states. This is readily apparent following a brief application of the partition function.

We begin by considering the average spin of a single lattice site [19],

$$\langle \sigma_i \rangle = \frac{1}{Z} \sum_{i}^{N} \sigma_i e^{-\beta E_i} \tag{12}$$

where the sum is taken by multiplying the possible spin states of σ_i by the Boltzmann weight, scaled by the corresponding energy of the system, E_i. The average alignment is then a simple extension given by,

$$\langle \sigma_i \sigma_{i+r} \rangle = \frac{1}{Z} \sum_{i}^{N} \sigma_i \sigma_{i+r} e^{-\beta E_i} \tag{13}$$

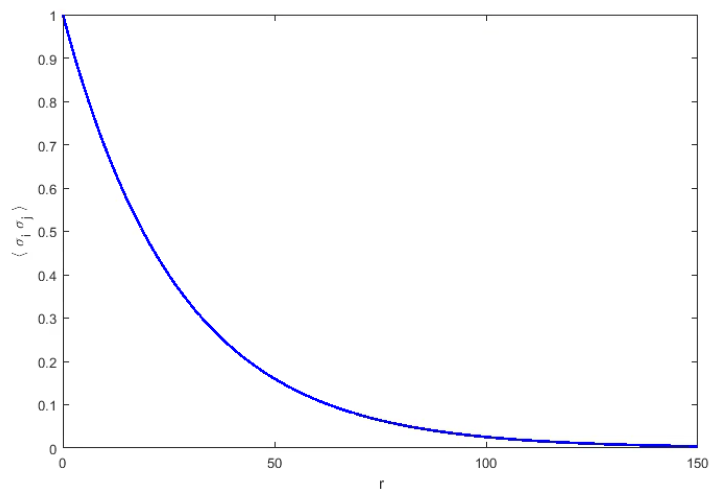

Figure 5. The spin correlation between two particles as the distance between the two, r, increases.

A cursory analysis of this relation reveals that as $r \to \infty$, $\langle \sigma_i \sigma_{i+r} \rangle$ goes to 0, as shown in Figure 5 [12]. This result, while built into the Ising model's exclusive concern with nearest neighbor interactions, is significant to our purposes.

If one were to imagine a legislation as an Ising chain, this enforced condition of locality allows the lattice to be populated in a manner which precludes interaction between certain legislators, particularly those which are opposed on ideological or partisan lines.

2.2 Kinetic Ising Models

Having studied the Ising model in equilibrium, we now turn to consider the model's non-equilibrium extension, *kinetic Ising models.* Kinetic Ising models are structured identically to Ising models, the critical distinction being an external agency induces spin flips within the system over time [20]. These spin flips are defined by a transition rate that leads to a steady state, which may in turn be studied using results from equilibrium statistical physics.

While the equilibrium properties of the Ising model follow clearly from the partition function and Boltzmann statistics, non-equilibrium properties depend on the nature of the system's spin dynamics

[12]. These dynamics, while often unique to particular systems, are flexible and numerically soluble so long as certain conditions are met. This flexibility in the dynamics is a cornerstone of non-equilibrium statistical physics, and demonstrative of the absence of the universal principles which dominate the study of equilibrium systems [21].

In order to simulate the Kinetic Ising model accurately, a modeled system must possess two critical properties, *ergodicity* and *detailed balance* [15]. Ergodicity refers to the capability of the system to reach any possible energy state from any arbitrary state. This condition implies that, however improbable, the system must be capable of eventually reaching any configuration with non-zero probability [13]. Given the Ising model's tendency towards equilibrium, or a steady state in the case of non-equilibrium models, this may seem impossible. However, the ergodicity condition is complemented by the detailed balance condition. This condition states that each transition of orientation is in equilibrium with its reverse process [13]. The detailed balance condition is essentially a statement of probability current conservation [21], given by

$$\omega_{s\to s'}P(s\to s') = \omega_{s'\to s}P(s'\to s) \tag{14}$$

Here s and s' are arbitrary states of the system, P is the probability of transition between any two states of the system, and ω are the transition rates. While perhaps a somewhat intuitive statement, this condition is critical in the study of non-equilibrium systems and the design of models which accurately describe those non-equilibrium systems. When taken together, the ergodicity and detailed balance conditions imply that Kinetic Ising model simulations will, eventually, reach a steady state, following the Ising models observed tendency to equilibrium [12].

Beyond satisfying these fundamental conditions, certain other considerations must be made in the construction of non-equilibrium spin dynamics. As we have said, equilibrium physics states that the equilibrium Boltzmann weights should be a solution of non-equilibrium dynamics [13]. Therefore, if a non-equilibrium model is meant to follow the Boltzmann distribution, $e^{\frac{\beta(E)}{Z}}$, the detailed balance condition states that

$$\frac{P(s')}{P(s)} = e^{\beta(\Delta E)} \tag{15}$$

That said, should a system begin far from equilibrium, other rates may also be selected in order to drive the system to equilibrium, or a steady state [13].

Furthermore, in order to better relate this system to the Ising model, the requirements of locality and symmetry should be taken into consideration [13]. Locality, as demonstrated by our treatment of the spin correlations, restricts dependencies of a single spin's transition rate to its nearest neighbors. Therefore transition rates are defined such that, $w_i(\sigma) = w_i(\sigma_{i-1}, \sigma_i, \sigma_{i+1})$. Complementing this condition, symmetry requires that the transition rate does not vary under a translational swap of nearest neighbors, $w_i(\sigma_{i-1}, \sigma_i, \sigma_{i+1}) = w_i(\sigma_{i+1}, \sigma_i, \sigma_{i-1})$.

In order to analyze a spin dynamic of particular interest to this project, which also satisfies the above conditions, we will now develop expressions for the equations of state and time dependencies of kinetic Ising models treated as being in contact with a heat reservoir, wherein a temperature dependency of spin states induce transitions as the system relaxes to equilibrium. We will assume that these spin transitions occur randomly but at a known rate. In order to solve the dynamics of these models, we turn to the Glauber method, also referred to in literature as Glauber dynamics.

Glauber dynamics, first proposed by Roy Glauber in 1963 [20], exactly solve the time-dependent behaviors for a one-dimensional Ising model in contact with a heat reservoir through both numerical and analytical methods and are capable of calculating the magnetization and two-site spin correlation functions exactly. Glauber dynamics are a single-flip kinetic generalization, meaning a single spin transition is considered at a time and as a result magnetization is often not conserved.

Kinetic Ising models solved according to Glauber dynamics are most completely described by the master equation, which expresses the conservation of probabilities for all configurations of a lattice of N sites [20],

$$\frac{d}{dt}p(\sigma_i, ...\sigma_N, t) = -\sum_i w_i(\sigma_i)p(\sigma_i, ...\sigma_N t) + \sum_i w_i(-\sigma_i)p(\sigma_i, ...\sigma_N, t) \tag{16}$$

where $p(\sigma_i, ...\sigma_N t)$ are the 2^N probability functions for all possible configurations, and $w_i(\sigma_i)$ is the transition rate of σ_i based on the fixed spins of all other sites. The sum is taken over all spins, considering the transfer of probability into configuration σ_i from other configurations in the gain term, or from σ_j into others in the a loss term.

The evolution of this configurational probability is dictated by a set of transition rates,

$$\sum_i w_i(\sigma_i)p(\sigma_i, ...\sigma_N t) = \sum_i w_i(-\sigma_i)p(\sigma_i, ...\sigma_N, t) \tag{17}$$

As has previously been stated, in order for these systems to resolve to a steady state, the detailed balance condition must be upheld. Therefore we select transition rates such that,

$$w_i(\sigma_i)p(\sigma_i, ...\sigma_N t) = w_i(-\sigma_i)p(\sigma_i, ...\sigma_N, t) \tag{18}$$

Under this condition, the probability currents are conserved in both directions between all possible configuration pairs. In order to relate the steady-state of this non-equilibrium system to the well-known Ising model solutions, Glauber dynamics enforces the condition that,

$$\frac{w_i(\sigma_i)p(\sigma_i, ...\sigma_N t)}{w_i(-\sigma_i)p(\sigma_i, ...\sigma_N, t)} = e^{\beta(\Delta E)} \tag{19}$$

Transition rates which satisfy this expression are then selected which further satisfy the locality and symmetry relations,

$$w_i(\sigma_i) = \frac{1}{2} - \frac{1}{4}\gamma_i\sigma_i(\sigma_{i-1} + \sigma_{i+1}) \tag{20}$$

Here the factor γ is related to the lattice site i by $\gamma_i = tanh(\frac{2J}{k_B T})$, which further relates the transition rate $w_i(\sigma_i)$ to the temperature of the heat reservoir.

These probabilities, $w_i(\sigma_i)$, can therefore be seen to take on three possible values for particles arranged in a non-periodic, one-dimensional lattice,

$$w_i(\sigma_i) = \begin{cases} \frac{1}{2} - \frac{1}{4}\gamma & \text{if } \sigma_{i-1} = \sigma_i = \sigma_{i+1} \\ \frac{1}{2} & \text{if } \sigma_{i-1} = -\sigma_{i+1} \\ \frac{1}{2} + \frac{1}{4}\gamma & \text{if } \sigma_{i-1} = -\sigma_i = \sigma_{i+1} \end{cases} \tag{21}$$

It is clear from these possible values alongside the Hamiltonian that so long as γ is positive, configurations of lower total energy will be more likely than configurations of higher energy [22]. As such, the factor γ is further defined such that, $0 \leq \gamma_i \leq 1$, in order to relate the behavior of the kinetic Ising model to the equilibrium model.

Having completely defined the spin dynamics of the kinetic Ising model, we now possess an expression for the time dependence of the probability of each possible spin configuration, as well as a means of solving for the time dependent behavior of macroscopic variables of interest. The magnetization functions are of particular interest here, given that they relate the alignment of particles over time, or, in an analogous legislative system, consensus over time.

The magnetization as a function of time is given as [22],

$$m = \langle \sigma_i \rangle = -\sum_{\sigma} \sigma_i p(\sigma_i, ...\sigma_N, t) \tag{22}$$

Taking this expression alongside the master equation, we see that the differential equations for the magnetization are therefore,

$$\frac{d}{dt}m_1 = -m_1 + \frac{\gamma}{2}m_2 \tag{23}$$

$$\frac{d}{dt}m_n = -m_n + \frac{\gamma}{2}(m_{n+1} + m_{n-1}) \tag{24}$$

$$\frac{d}{dt}m_N = -m_N + \frac{\gamma}{2}m_{N-1} \tag{25}$$

Having considered the Ising model both in and out of equilibrium, we now turn to the final extension necessary to this thesis, the multi-temperature Ising model. This model may be solved by a generalization of Glauber dynamics provided by Racz and Zia [22]. In this case, each lattice site, n, within the Ising chain is in contact with heat baths of various temperatures, T_n. If T_n is different for any two lattice sites within N, the system is perpetually unable to reach equilibrium as each unique heat bath drives the system toward a different equilibrium. Therefore, energy flows between lattice sites at different temperatures. While the system will reach a steady state under these conditions, it will not be an equilibrium state.

Under these conditions, the equation of states and time dependencies are identical to those for the single-temperature kinetic Ising system; however, the possible variance in temperature at each lattice site significantly complicates solutions, due to the temperature dependency of the factor γ. As such, γ now takes a unique value for each lattice site i, given by $\gamma_i = tanh(\frac{2J}{k_B T_i})$.

In the multi-temperature case, the magnetization takes the form of a matrix equation comprised of real, non-negative elements [14],

$$M = \begin{bmatrix} -1 & \frac{\gamma_1}{2} & 0 & \dots & \frac{\gamma_1}{2} \\ \frac{\gamma_2}{2} & -1 & \frac{\gamma_2}{2} & \dots & \\ & \frac{\gamma_3}{2} & -1 & \frac{\gamma_3}{2} & \dots \\ & \vdots & \ddots & & \\ \frac{\gamma_N}{2} & \dots & \dots & \frac{\gamma_N}{2} & -1 \end{bmatrix} \tag{26}$$

This matrix can then be simplified by the transformation $m_n \to \frac{m_n}{\sqrt{\gamma_n}}$, giving the symmetric matrix [14],

$$M = \begin{bmatrix} -1 & \frac{\sqrt{\gamma_1\gamma_2}}{2} & 0 & \dots & 0 \\ \frac{\sqrt{\gamma 1\gamma_2}}{2} & -1 & \frac{\sqrt{\gamma_2\gamma 3}}{2} & \dots & \\ 0 & \frac{\sqrt{\gamma_2\gamma_3}}{2} & -1 & \frac{\sqrt{\gamma_3\gamma_4}}{2} & \dots \\ & \vdots & \ddots & & \\ 0 & \dots & \dots & \frac{\sqrt{\gamma_{N-1}\gamma_N}}{2} & -1 \end{bmatrix} \tag{27}$$

While this final extension in the development of the multi-temperature kinetic Ising model does complicate matters, it should be stated that this provides a further opportunity to develop more complex and dynamic models. These further dependencies and considerations which must be made in developing complete descriptions of non-equilibrium systems, while perhaps increasing computational load, provide more nuanced dynamics in each model, thereby revealing more insightful correlations.

2.3 Methods of Solving Kinetic Ising Models

While the Glauber method is of particular interest to this thesis, dynamics of one-dimensional spin systems out of equilibrium may be also simulated and solved by a variety of stochastic algorithms, each informed by their own mathematical machinery and analogues to physical features. Moreover, as it is this thesis's goal to elaborate the variety of methods through which statistical physics might be applied in the simulation of non-physical systems, a foray into these numerical methods is worthwhile. Therefore we will conclude our exploration of the Ising model here as we consider the translation between method and algorithm, by describing alternative methods to Glauber dynamics such as the Metropolis and Kawasaki methods.

Numerical solutions to kinetic Ising models are simulated through an algorithm based on Markov Chain Monte Carlo (MCMC) methods [23]. The Markov chain is an algorithm for describing stochastic processes, which generates a sequence of outputs based solely on a previous output. Markov chains are therefore 'memory-less,' meaning that all information necessary to predict the next output is present in the current state [24]; this is referred to as the Markov Property. MCMC methods construct multiple Markov chains from a continuous random variable whose values are accepted or rejected with a probability density proportional to a known function. In the case of one-dimensional spin systems out of equilibrium, the random variable is the energy change of the system as a result of a randomly selected spin flip, while the function is the Boltzmann distribution. As these chains develop, their sum therefore tends towards a similar proportionality with the known function. As a result, the Markov chains are generated such that the equilibrium distribution of the chains is similarly proportional to the known function [23]. Through this algorithm, MCMC methods approximate the posterior distribution of a parameter of interest by random sampling in a probabilistic space [24].

The Metropolis method is likely the simplest of Ising model simulation algorithms, and, similarly to Glauber dynamics, is a 'spin-flip' algorithm, meaning it induces a change in orientation for a single spin at a time [13]. Under this algorithm, the probability with which the system moves between configurations at each update event is given by [25],

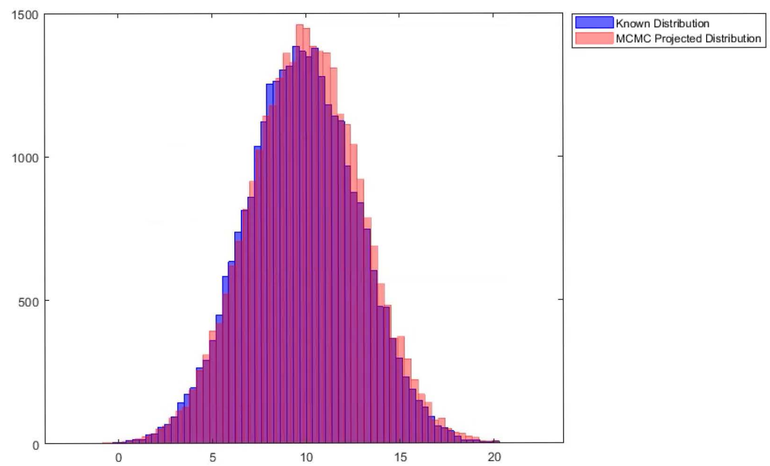

Figure 6. A comparison of a given, arbitrary probability distribution and the approximated posterior distribution generated by an Markov Chain Monte Carlo algorithm.

$$P(s \rightarrow s') = g(s \rightarrow s')A(s \rightarrow s') \tag{28}$$

where $g(s \rightarrow s')$ is the selection probability and $A(s \rightarrow s')$ is the acceptance ratio. The selection probability indicates which configurations can be reached by the algorithm from a preceding configuration. The acceptance ratio is the fraction of events in which the transition takes place.

The Metropolis method assumes that the acceptance ratio, A, is given by [25],

$$A(s \rightarrow s') = \begin{cases} e^{-\beta \Delta E} & \text{if } \Delta E > 0 \\ 1 & \text{if } otherwise \end{cases} \tag{29}$$

Therefore, at each update event, any transition which lowers the energy of the system is accepted, while any transition which raises the energy of the system is accepted with a probability proportional to the Boltzmann weight associated with the energy difference between the initial and final state.

While many simulations of the Ising model are 'spin-flip' algorithms, the Kawasaki method simulates Kinetic Ising models by swapping the orientation of spins [23]. As such, the Kawasaki method is a magnetization conservation method, meaning the total magnetization is unchanged over each iteration of the algorithm. This is useful in studying systems dominated by a single magnetization phase; while a given system may tend to positive magnetization, the Metropolis method is equally as likely to select and settle to a negative magnetization. The Kawasaki method comes in two varieties, the local and nonlocal Kawasaki methods. The local Kawasaki method refers to algorithms which swap only neighboring spins. The non-local Kawasaki method swaps spins regardless of their separation along the lattice. While the non-local is more efficient, meaning simulations tend towards equilibrium with less computation time [25], it is a non-physical simplification as spin transitions therefore do not depend on the physical distance between coupled particles, thereby disregarding the locality requirement implicit in most Ising simulations. Outside of this modification, the Kawasaki method and Metropolis method are largely identical. Specifically, the acceptance ratio remains the same as that selected for the Metropolis method.

While these methods for simulating and thereby solving kinetic Ising models over discrete time steps are versatile and precise, we have shown above in our treatment of the system's magnetization that the Glauber master equation can be easily manipulated to form efficient expressions for the system's time dependent functions of state. As such, the ODEINT routine in the Python package SCIPY is an easily accessible means of simulating the dynamics of the system. In this case, a system simply evolves according to the selected transition rates on an arbitrary time scale, within which the integrated magnetization equations are solved at each discrete time step [22].

2.4 Network Theory and Social Networks

The second class of models we employ are weighted, complex networks. Network theory, whose first relation was proven in 1736 by Euler's solution to the Seven Bridges of Könisberg problem [26], is the analysis of graphs as representations of symmetric and asymmetric relations. Complex network theory, by extension, is the analysis of interacting nodes represented within a graph. While a complete overview of network theory is outside the scope of this project, given that the mathematical and computational machinery of each network is at least somewhat unique to the systems analyzed, a brief overview of terms and analytical techniques related to network theory is worthwhile here.

A network, in the simplest terms, is a two-dimensional representation of nodes and edges which connect those nodes [30]. A network is depicted as a graph, defined as the ordered triple $G = (V, E, f)$ wherein $V = \{v_1, v_2, ..., v_n\}$ is the set of nodes and $E \subseteq V \times V$ is the set of edges [27]. We define n and m as the cardinality of the sets V and E respectively, such that, $\mid V \mid= n$ and $\mid E \mid= m$. The function f is then the function which maps elements of E to corresponding pairs of elements in V.

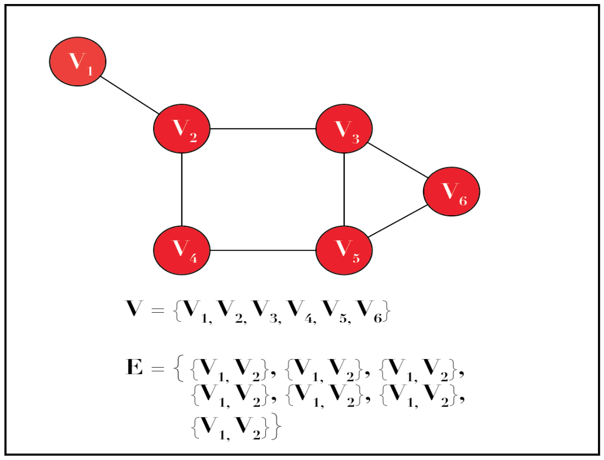

Figure 7. A simple network in both graphical and mathematical terms.

Within a graph, edges may be either *directed* or *undirected*. Undirected edges denote a binary, symmetric relation between nodes, while directed edges denote the relationship between a node to another, which may not be symmetric [28]. A graph is therefore undirected if, for any two pairs of vertices, $(v, u) \in E$ and $(u, v) \in E$. The degree, d_i, of a node, v_i, is the number of edges incident upon the node, and is given by the degree value, k_i. The degree sequence is defined at, $\mathbf{d}^T = [d_1 d_2 d_3 \ldots d_n]$ [27]. The function, f, may also determine the edge weight and length of connections made [28]. The dependencies of this function vary with application, though typically within the set V, nodes may be assigned any number of items of information significant to their graphical position, propensity to form edges, etc [28]. A simple graph refers to a graph wherein nodes are connected without multiple edge between the same nodes or loops (edges which connect the node to itself) [27]. We will consider exclusively simple graphs for the purposes of this thesis.

That said, various other descriptors will be significant to our model. A graph may be described as connected if any node can be reached by traveling along the edges starting with any given node [29]. A graph may also be strongly connected if the edges provide a direct path between any two nodes [10]. The maximum number of edges possible in an undirected graph of N nodes is $\frac{N(N-1)}{2}$. Furthermore, a graph may be described as k-connected if each node has the same degree, k. A given graph may also be broken into subgraphs, or a set of nodes connected by edges which may be extracted and analyzed independently of the graph itself [30].

Of particular note here are networks of social influence and interaction, specifically voter models and models which may be adapted to simulate voting behaviors. In general, our attention is focused on connected graphs comprised of nodes, encoded into which is an index of data which determines that nodes position, capability to influence other nodes, and susceptibility to influence from other nodes. For the purposes of this thesis, we will refer to those nodes as agents.

In the field of network theory, significant work has been undertaken to analyze the effect of different structural and topological network parameters on the evolution of cooperation [10]. Significant parameters include degree distribution, degree heterogeneity, average degree, and assortativity [29].

Degree heterogeneity is a relatively simple concept. A network may be considered homogenous or regular if any given node may be reached in the same number of steps from any other node [28]. Conversely, degree heterogeneity refers to the spectrum of degree values within the network [28]. Furthermore, degree heterogeneity provides a measure of a network's diversity of structures. As the spectrum of d-values of the nodes increases, the network becomes more irregular and complex. This spectrum is measured through the degree distribution, $P(k)$, which determines how many nodes in a network have the given degree, k. For example, in stochastic networks in which edges are generated randomly but with a given probability, $P(k)$ is a Poisson distribution around the average degree, $\langle k \rangle$ [10].

The first proposed measure of heterogeneity comes from Snijders, alongside a modification by Bell [28], is given as

$$P(k) = \frac{1}{N}\sum_{i}^{N}(k_i - \langle k \rangle)^2 \tag{30}$$

This expression is particularly accurate in the case of random graphs [10], and is sufficient to our purposes.

That said, the development of network edges may either be random or preferential to varying degrees; this is referred to as assortativity. For the purposes of a social network, these preferences usually depend on either following a probability distribution which favors nodes of higher degree or on certain criteria nested within each node in the set V [27]. We will proceed with the latter definition, as it most closely relates to the tendency of politicians to interact with members of the same political party.

Closely related to measurements of interactions between agents of differing political parties, and especially pertinent to the needs of this thesis, are measurements of cooperation. We will proceed with the definition, provided by Krackhardt and Stern [31], that cooperation index, μ is given by,

$$\mu = \frac{E_I - E_E}{E_I + E_E} \tag{31}$$

where E_I is the number of edges internal to an organization, while E_E is the number of edges exiting an organization.

These fundamentals of network theory alongside these metrics of graphical qualities form the foundation by which social networks may be constructed and analyzed. We apply these in the context of the United States legislature in section 4.

3 Kinetic Ising Model of the United States Legislature

We now present a kinetic Ising-type model of the United States Legislature. This model is constructed in Python 3.7, and the code is provided in Appendix A. Within this model, each lattice site of a finite Ising chain is populated by a two-spin particle, representing a legislator, in either the 'yes' state ($s = 1$) or 'no' state ($s = -1$). The current state of each particle represents that legislator's vote.

The sites are arranged to reflect the spectrum of political ideologies, such that s_1 and s_N may be considered the ideological extremes. As such, the nearest neighbors of each particle represent the most ideologically similar legislators. The Ising chain is considered to have non-periodic boundary conditions in order to reflect the lack of influence between ideologically opposed legislators. As such, given the locality condition, spin transition interactions occur only between ideologically similar legislators.

The initial spins of the particles at each site are set to either alternate between the two possible spin states (i.e. $s_1 = 1, s_2 = -1, s_3 = 1 \ldots$), or be evenly distributed between portions of the chain. Therefore, the initial magnetization is 0.

Equal sections of the Ising chain are treated as being in contact with a heat reservoir, with portions segmented according to ideological similarity. The temperature of each heat reservoir is assigned, and while the temperature is an arbitrary value, it critically informs the dynamics of the system.

We select the standard transition rates described by Glauber, alongside the multi-temperature extension from Racz and Zia. As a result, we express the magnetization of each sub-lattice according to the differential equations provided above,

$$\frac{d}{dt}m_1 = -m_1 + \frac{\gamma_n}{2}m_2 \tag{32}$$

$$\frac{d}{dt}m_n = -m_n + \frac{\gamma_n}{2}(m_{n+1} + m_{n-1}) \tag{33}$$

$$\frac{d}{dt}m_N = -m_N + \frac{\gamma_n}{2}m_{N-1} \tag{34}$$

From these initial conditions, the magnetization of each particle is solved using Python package SCIPY routine ODEINT, which solves the master equation and differential equations of magnetization for individual particles. These values are then stored, and total magnetization of the system is calculated. The total magnetization of the system represents the average alignment of particles in either the 'yes' or 'no' states. As a result, the total magnetization represents the consensus of the entire lattice, while the magnetization of each sub-lattice is the consensus of each ideologically-separated population. A positive magnetization indicates a majority in the 'yes' state, while a negative magnetization indicates a majority in the 'no' state.

In order to apply this model to the United States Legislature, we consider an Ising chain of 435 sites, representing the House of Representatives. Each site is occupied by a two-spin particle, representing a voting member of the House of Representatives. Lattice sites are arranged according to political ideology, such that sites $n = 1$ through $n = \frac{N}{2}$ may be considered as representing the comparably more 'liberal' legislators and $n = \frac{N}{2} + 1$ through $n = N$ may be considered the comparably more 'conservative' legislators. We divide this Ising chain into equal sub-lattices, each treated as being in contact with a heat bath.

We begin by dividing the chain into three sub-lattices; the results of this simulation are provided in Figure 8. We find that regardless of initial conditions the system eventually moves towards a steady state in which total magnetization is 0. This indicates that the system inevitably fails to reach any consensus, as an equal number of legislators are in the 'yes' and 'no' states. In order to examine this system further, we observe the localized magnetization of each sub-lattice, provided in Figure 9.

We note that the magnetization of the system tends to follow the behavior of those regions in contact with the lowest temperature heat reservoir, as those regions move most slowly to the steady state. This follows naturally from the model's parameters under Glauber dynamics. Due to the temperature dependency of γ in the magnetization differential equations, temperature could be treated

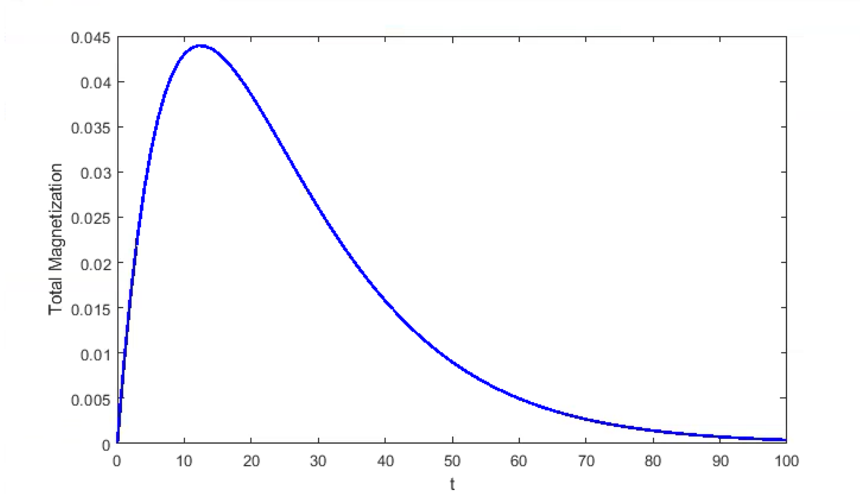

Figure 8. The change in overall magnetization for a lattice chain of length $N = 435$, wherein three equally segmented populations are in contact with a corresponding heat bath of various temperatures. The time scale is arbitrary.

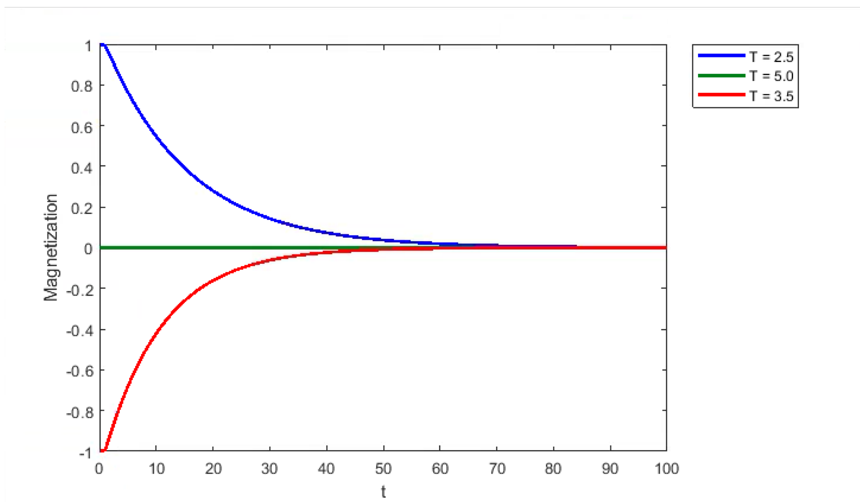

Figure 9. The change in overall magnetization for a lattice chain of length $N = 435$, wherein three equally segmented populations are in contact with a corresponding heat bath of various temperatures. The time scale is arbitrary.

as the tendency of each sub-lattice's population of particles to align with the system. As such, the higher the relative temperature of the reservoir, the faster the system will move towards the steady state; conversely lower temperatures relate a lower affinity towards alignment, or 'stubbornness'.

This result is better demonstrated in the case in which the Ising chain is divided into five equal sub-lattices, each in contact with a heat reservoir of a distinct temperature. Here we observe that the magnetization of any two sub-lattices, set to identical initial spin configurations, will move towards

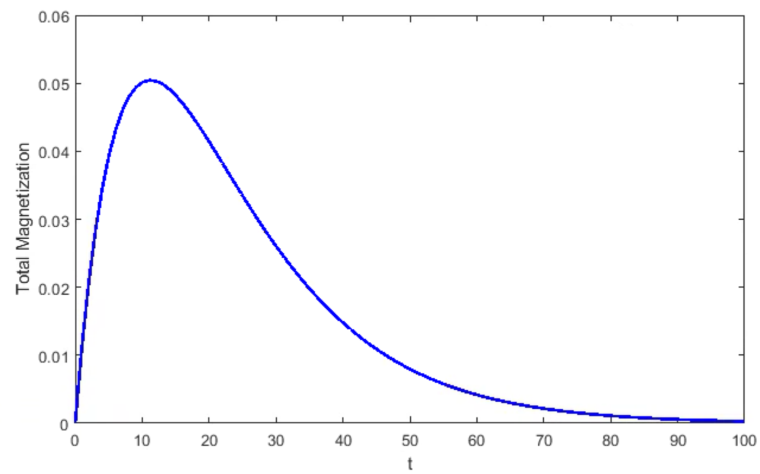

Figure 10. The change in overall magnetization for a lattice chain of length $N = 435$, wherein five equally segmented populations are in contact with a corresponding heat bath of various temperatures. The time scale is arbitrary.

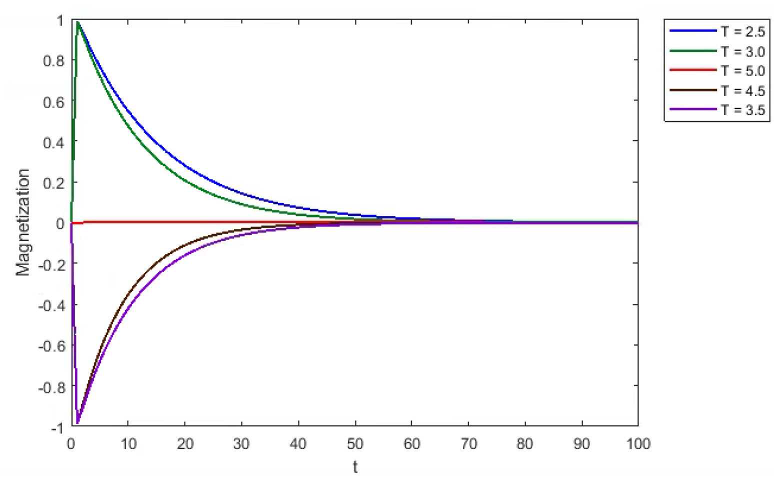

Figure 11. The change in individual magnetizations for five populations occupying equal portions of a lattice chain of length $N = 435$. The lattice sites comprising each population are in contact with a corresponding heat bath of varying temperatures. The time scale is arbitrary.

the steady state at different rates which are inversely proportional to temperature, as in the case of the sub-lattices in contact with heat reservoirs $T = 2.5$ and $T = 3.0$ shown in Figure 11. The total magnetization of the five temperature system is provided in Figure 10, and the localized magnetization for each sub-lattice is provided in Figure 11.

When considered in the context of the United States Congress, this model presents an immediately apparent shortcoming. While this model invariably predicts that the House of Representatives will not be able to reach a consensus regardless of their initial vote or stubbornness, approximately 3% of bills were enacted into law under the 115^{th} Congress [32]. Therefore, an accurate predictive model should allow for at least certain cases to yield a positive consensus. That being said, the model does reflect the lack of compromise between ideologically opposed legislators frequently criticized in the current field of American politics.

Moreover, despite the consistent tendency of this model to move towards a steady state of 0 magnetization regardless of initial conditions, certain lattice sites and graphical features are worth exploring further. Most notably, the particles occupying lattice sites $n = 1$ and $n = N$ display noteworthy behaviors. As a result of the non-periodic boundary conditions within which the model is constructed, these particles possess one less nearest neighbor interaction than other particles. Moreover, these particles occupy lattice sites maximally removed from those sub-lattices in contact with other heat reservoirs. As a result, the local interactions which define the Ising model, as well as the flow of thermal energy which drive the kinetic Ising model, are minimally expressed at these sites. Consequently, these sites resist any transition to a non-zero steady state within the sub-lattice population, and as such rapidly contribute to the system's move towards its steady state of 0 magnetization. This provides what is perhaps this model's most critical insight on voting behavior. Specifically, that ideological extremism limits cooperation.

Beyond these particular sites, the short-term behavior of the system is also noteworthy. Within the short-term, the total magnetization of the system does have a non-zero magnitude. This indicates that early in the simulation, a majority of particles occupying the Ising chain are in a single state. Given that legislation brought before Congress is rarely voted on over as many iterations as we present here, this is certainly worth noting.

There are several avenues by which to improve this model in order to better reflect the United States Congress and potentially yield more significant results. While the unrealistic time-scale in which these simulations take place has already been noted, this is not a discrepancy within the model itself, but rather a consideration on the significance of certain data. Otherwise, the model could incorporate the already well-defined features of the multi-temperature kinetic Ising model. While the influence of ideologically similar legislators is a significant factor in determining a legislator's vote, external influences are abundant. The behavior of this multi-temperature kinetic Ising model in an external magnetic field may be considered in order to simulate the effects of lobbying groups, constituents, or the media. Moreover, as the political agenda and opinions of these groups are varied (and as such the politicians they appeal to are varied), this magnetic field could be treated as being dependent upon the heat reservoir in contact with each sub-lattice, allowing for a variety of magnetic fields to impact each ideologically separated sub-lattice in unique ways. The spin correlation functions may also be considered for particles occupying lattice sites at the borders of the sub-lattices in order to analyze the impact of ideologically similar particles in contact with distinct heat reservoirs.

4 Social Network Model of the United States Legislature

We now present a model which applies network theory in order to simulate the voting behaviors of the United States Legislature. This model was constructed in Python 3.7, and the code is provided in Appendix B. This model is particularly concerned with the effect of variances in degree heterogeneity, average degree, and assortativity on agent's cooperation and capability to resolve to a steady state.

Within this model, each agent is assigned a political ideology value, α_i defined such that $0 \leq \alpha \leq 1$. A political ideology value of 0 indicates that the agent is maximally liberally inclined, while a political ideology value of 1 indicates that the agent is maximally conservatively inclined. Each agent is capable of existing in either the 'yes' state, ($\sigma_i = 1$) or 'no' state ($\sigma_i = -1$), which represents their current vote.

The name, political party, and political ideology value of each agent is stored in a PANDAS data-frame. An issue value, z, then is selected in order to represent each agent's affinity for a legislative issue, and is defined such that $0 \leq z \leq 1$. Following the introduction of the issue value, the initial vote of each agent is also determined and stored within the data-frame. The initial vote is selected by the NUMPY package's random number routine with the probability of a 'yes' vote given by,

$$P_i = 1 - \mid \alpha_i - z \mid \tag{35}$$

This probability is selected in order to maximize the probability of a 'yes' vote in the case that $\alpha_i = z$.

Critically, each agent's initial vote is formed in a vacuum, without any external influence from other agents. At this point, however, edges are randomly placed between agents, with one edge formed per agent. These edges represent the connected agents' ability to influence one another in order to align their votes. These edges are considered undirected, under the assumption that these social influences are inherently mutual.

These edges are then weighted according to each agent's ability to influence the vote of the other. In order to identify which forms of social influence are significant enough to contribute to this process, we turn to 3 of the canonical mechanisms of social cooperation [33], which have clear political analogues:

1 Kin selection, or an agent's predisposition to cooperate with others in a shared population, in this case a shared party affiliation.

2 Direct reciprocity, or an agent's predisposition to cooperate in order to gain some obvious benefit from those they cooperate with, in this case an agent's willingness to vote alongside agent's of a similar political learning with the expectation of ongoing political support.

3 Network reciprocity, or an agent's predisposition to cooperate with others they are closely affiliated with, in this case frequent interactions as a result of caucuses, committees, etc.

In order to quantify these mechanisms, we define the edge weight between two agents, w_{ij}, as,

$$w_{ij} = \gamma(1- \mid \alpha_i - \alpha_j \mid) + \beta \tag{36}$$

where γ represents the frequency of interaction between agent's, and β is a boolean value of either 1 in the case that connected agents share a party affiliation or -1 in the case that they do not.

The nested function, $1- \mid \alpha_i - \alpha_j \mid$, is selected in order to maximize agent's influence in the case that $\alpha_i = \alpha_j$.

The magnitude of all edge weights connected to an agent are then summed in order to identify the influence, ρ, of the agent's connections on their voting state, with the final value given by,

$$\rho = \sum_{(ij) \in E} -\frac{\sigma_i}{\sigma j} w_{ij} \tag{37}$$

If connected agents' initial votes are out of alignment, this contributes positively to the sum, as $\frac{\sigma_i}{\sigma_j} = -1$. If connected agent's initial votes are aligned, this contributes negatively, as $\frac{\sigma_i}{\sigma_j} = 1$. If ρ is greater than P_i, the agent's initial affinity for the issue, then the influenced agent changes their vote; otherwise, their vote remains unchanged.

The process of assessing this influence and consequently changing votes is a single voting step in this model. The critical application of this model is found in carrying out multiple iterations. At each voting step one new edge per agent is formed. As such, the average degree necessarily increases by 1 for each iteration, and the network will be completely connected following $\frac{N-1}{2}$ voting step iterations. The heterogeneity of the network, given that edges are formed randomly, conforms closely to the Poisson distribution [28].

The probability with which edges are formed between members of the same party or ideologically similar agents can also be adjusted, thereby disrupting the degree heterogeneity and increasing the assortativity of the network. Moreover, throughout each step of the algorithm, a permanent vote can be assigned to agents. This freezes the agent in a particular state in order to simulate the effect of maverick voting. Similarly, in order to simulate bloc voting, multiple agents can be assigned the maverick state, as well as a shared and unchanging initial vote.

The introduction of non-cooperative voting and preferential connection in this voting network has unique and interesting results on both the behavior of the model.

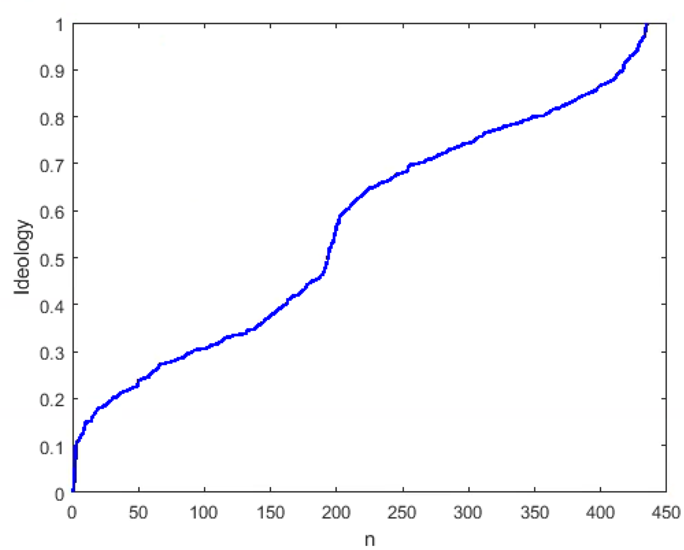

Figure 12. The spread of political ideologies, α, within the network.

In order to apply this model to the United States Congress, we populate this network with 435 agents, representing legislators in the House of Representatives. Within each node the political ideology and party affiliation of a corresponding legislator from the 115$^{\text{th}}$ United States Congress is stored, with values taken from GovTrack's 2017 'Report Cards' [5]. Within this population, 238 agents belong to the Republican Party while 197 agents belong to the Democratic Party. We note that within this system, a disproportionate amount of agents possess ideologies at or near both the extreme and median values. The spread of political ideologies are provided in Figure 12.

We first consider a scenario wherein agents form edges completely randomly, with no bias towards party affiliation or political ideology. We observe that presented with any issue value the total votes will eventually collapse to a complete consensus, with the total number of agent's in the 'yes' state moving to 0 or 435. The sum of agents in the 'yes' state over each iterative step under these conditions is provided in Figure 13.

In this case, the system moves towards a consensus aligning with the average initial vote of the political party whose cooperation index is initially greatest, meaning agents within the party formed the most external connections during the initial iterations. Given that these connections are formed randomly, the likelihood of completely aligning in the 'yes' or 'no' state is largely random and independent of the issue value.

Of particular note here are agents whose ideological values are near the median, $\alpha \approx 0.5$, and whose initial vote reflects the average state of their political party. These agents show an especially strong influence on agents belonging to the opposing political party. This follows naturally from the model, as their neutral ideological value maximizes their potential influence on agents of other parties.

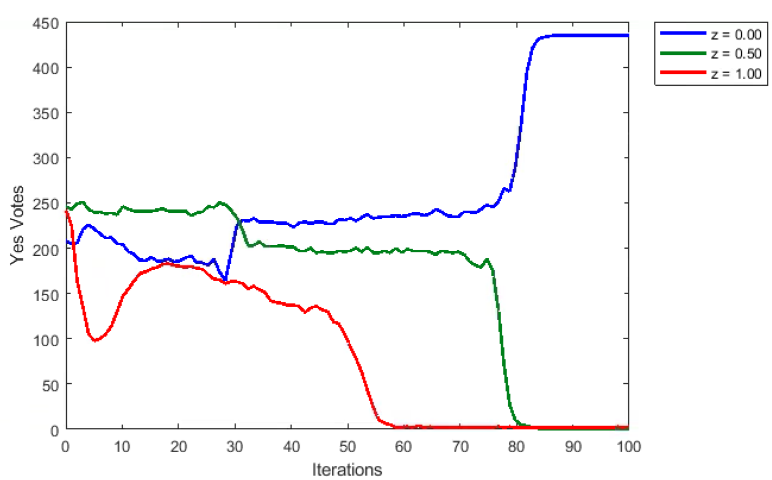

Figure 13. The sum of all agents in the 'yes' state over multiple iterations.

We then consider the case in which politicians of the same political party prioritize connection with one another. In this case, the network in the initial iterations can be effectively considered two distinct subgraphs, each populated by one of the two political parties. As the maximum number of edges is approached for each subgraph, $\frac{N(N-1)}{2}$, the subgraphs then begin to interact. Critically, as the Democratic party is occupied by less agents, Democratic agents will tend to find connections outside their political party first.

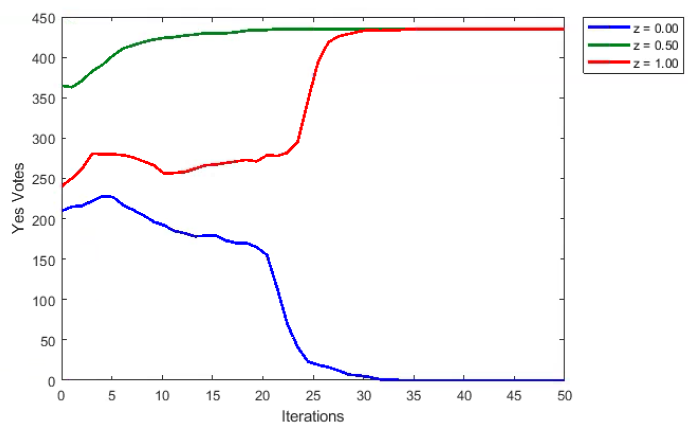

Figure 14. The sum of all agent's in the 'yes' state over multiple iterations, throughout which agents formed networks with a bias towards shared political party.

The sum of 'yes' votes over each iterative step under these conditions is provided in Figure 14. When the agents connect preferentially, we observe that the system still resolves to consensus regardless of the issue value presented. However, the consensus reached is no longer randomly dictated by the most cooperative party. Under these conditions, voting states on conservative issue values, $0.75 \leq z \leq 1.0$ tend to resolve to the 'yes' state, as Republican voters are more likely to initially vote, 'yes' on conservative issues. Conversely, voting states on liberal issue values, $0.0 \leq z \leq 0.25$ tend to resolve to the 'no' state. Neutral or middle of the line issue values continue to yield a random consensus to either state, which is determined by the average initial voting state of the system. These results are to be expected, given that Republican conservative voters outnumber Democratic liberal voters. As each party forms connections with like-minded agents over initial iterations, they quickly come to a consensus. As Democrats run out of party members to connect with, the Republicans are still able to connect with like-minded agents. As such, when Democrats eventually begin to connect with Republican agents, their individual influences cannot overcome the influence of the party. This rapid collapse to consensus within the partisan subsystem also accounts for the notably smaller amount of voting steps required to reach a consensus on the issue.

We further observe that agents whose ideological values are near the median are no longer significantly influential within the system. Rather, agents at the conservative ideological extreme, $z \geq 0.9$, tend to determine the final consensus.

With this result in mind, we next enforce the condition that all agents whose ideological values are at the political extremes, $\alpha \leq 0.1$ or $\alpha \geq 0.9$, are unable to change their initial vote. The sum of agents in the 'yes' state over each iterative step under these conditions is provided in Figure 15. Under these conditions, the system is incapable of reaching a complete consensus, as a number of agents will not change states regardless of influence. That being said, we once again note that given a near neutral issue value, the system will approach a consensus. This consensus is once again randomly determined based on the average initial vote of the system. For conservative and liberal issue values, the system is unable to reach a consensus over any number of iterations. After the network is strongly connected over initial iterations, the total votes tend to oscillate as frozen agents induce changes, which are then reversed by frozen agents on the other end of the political spectrum.

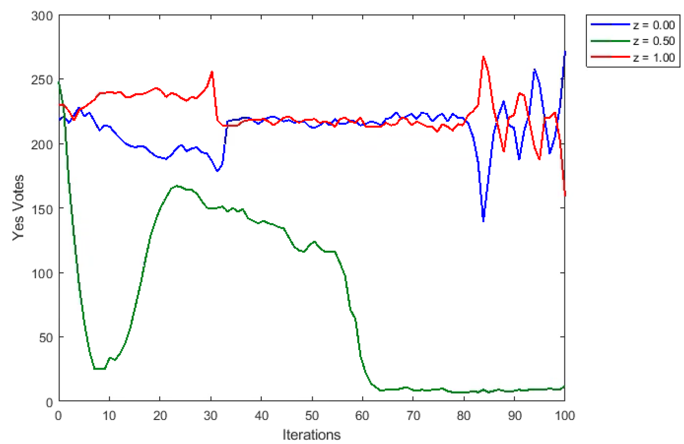

Figure 15. The sum of all agent's in the 'yes' state over multiple iterations, throughout which the votes of agents with ideologies $\alpha \leq 0.1$ and $\alpha \geq 0.9$.

When considered in the context of the United States Congress, this model possesses a noteworthy inaccuracy. Regardless of preferential connection or bloc and maverick voting, this model predicts complete consensus on certain legislation. As has already been noted in this thesis, within the roll call vote which this model emulates, this occurs with a significant statistical infrequency in the House of Representatives.

That being said, the insights into cooperation between political agents this model provides are not insignificant. From initial trials on random connections between agents, we observe a party's willingness to cooperate with the opposition leads to consensus in that party's favor. Moreover, we observe that centrist agents are able to generate bi-partisan support.

When connections are made preferentially along partisan lines, we observe that cooperation between parties is considered secondary to generating a party-wide consensus. As a result, the majority party is able to enforce their consensus on the system. Finally, in the case of bloc and maverick voting, we observe that an unwillingness to cooperate drastically affects the system's ability to resolve to consensus, eventually leading to a political tug of war between unyielding legislators. Taking these insights into account, we note that this model leads to the crucial, if somewhat obvious, conclusion that bi-partisan interaction, centrist politics, and a willingness to yield a position under social influence should naturally yield a legislative consensus.

There are several potential avenues through which to improve this model in order to make it not only more reflective of the United States Congress, but also to potentially yield more significant results. As in the case of the multi-temperature kinetic Ising model, the most obvious consideration to be made is on the model's iterative nature. Legislation in Congress is typically voted on once or, occasionally, twice. While the same legislation may be presented multiple times following amendments or be placed inside a larger omnibus bill, it would be inaccurate to ascribe those amended or compiled bills an identical issue value. As such, we might consider not only adding connections between agents at each iterative step, but also introducing a separate issue value for each iterative step.

Furthermore, legislation is not introduced directly to the entirety of a Congressional body. Legislation is first presented in committees, wherein anywhere from 12 to 63 legislators must vote on the bill before it is presented to Congress. While only those legislators within the committee vote on the bill in question, it would be inaccurate to say that social interactions and influences which decide their final votes are limited to those committee members. Therefore, we might consider either scaling the size of the network to consider only the votes and interactions of legislators within the committee, or examining a committee as a cluster within the network.

Finally, the factors which influence legislators cannot be limited to exclusively their fellow legislators. Constituents, lobbyists, and media attention are significant influences as well. Therefore, we might consider incorporating these influences into the model. This could be accomplished by connecting each agent to a node which does not contribute to the sum of votes or change its vote, and assigning that node an ideological value representing the average political stance of that agent's electorate. While this may be difficult to implement with accuracy in respect to the actual political mindset of a given state or congressional district, a general linear progression from 0 to 1 based on registered voters may prove sufficient to provide interesting results. Finally, GovTrack releases leadership values for each legislator, calculated according to that legislator's influence on other voters. We might consider making the edges of the network directed and scaling influence between legislators by that leadership value.

5 Conclusions and Further Applications

In this thesis we have presented two models of the United States Legislature, alongside the body of statistical physics and network theory which both inform and define them. We then applied those models specifically to the United States House of Representatives. These have provided distinct, yet complementary results, as well as interesting insights into the United States political system.

From both the multi-temperature kinetic Ising model and social network model we observe that politicians on the extremes of the ideological spectrum may significantly impact cooperation within a system. Either driving the system towards a lack of consensus, as in the case of the kinetic Ising model, or enforcing their unpopular opinion on otherwise cooperative voters, as in the case of bloc voting. We further note that the kinetic Ising model seems to indicate that polarization between ideologically separated groups seems inevitable, a dire conclusion, though one that resonates with the current perception of American politics. However, we also observe that communication between ideologically divided political parties does yield consensus in the case of the social network model, a reassuring thought following an examination of the kinetic Ising model. In either model, we see that cooperation between ideologically opposed organizations is critical to reaching a consensus. While this is perhaps an obvious result, given the sociological conundrum that is cooperation, expressing this result under such simple two-state parameters is worthwhile.

That being said, the complexity of American politics goes beyond the scope of this thesis, and is undeniably somewhat random given its dependence on human behavior. Traditionally 'liberal' issues may appeal to a 'conservative' legislator and vice versa for a variety of reasons including personal bias, lobbying influences, or the particular benefit the issue in question provides to that legislator's constituents. Given the extreme difficulty which would accompany quantifying these factors in a legislator's decision, it should not be surprising that there is a lack of reputable research on the topic.

As such, the results of these models, while certainly interesting, could not be considered predictive or truly reflective of political realities as of yet. Regardless, the field of socio-physics continues to grow, and alongside it legislative forecasting models grow more complex and nuanced [34] [35] [36] [37].

With that in mind, we present this thesis as a contribution to the ongoing effort to bridge the divide between physical and social sciences, and the pedagogical and analytical role of statistical physics in that confluence. Moreover, given the range and versatility of non-equilibrium statistical physics and network theory, this thesis presents these models not only as a potential means of studying American politics, but also to demonstrate the flexibility in applying these models to study a variety of systems.

While the Ising model's relevance in legislative forecasting efforts is clearly its most significant contribution to this thesis, it should be noted that the Ising model has been applied in a variety of other non-physical contexts. Among the most conceptually diverse of these applications include studies of tree yield in timberland [38], rumor propagation in confined populations [39], and cancer growth in isotropically arranged cells [40].

Similarly, social network models enjoy a diverse range of applications. Given their topological and mathematical flexibility, these networks may be adapted to reflect a variety of complex networks. These include ecological models which predict the impact of invasive species [41], the evolution of game theory strategy in the canonical Prisoner's Dilemma [42], and, perhaps most significantly in recent times, the spread of pandemic infections under various social parameters [43].

Each of these models may provide not only valuable data on the dynamics and dependencies of a system, but also a conceptually clear educational tool for translating complex biological, sociological, or ecological information to the public through these models' readily apparent results and graphical representations. Moreover, these systems provide a valuable pedagogical tool, and in the information age, wherein knowledge has become increasingly democratized, this is a worthwhile and valuable endeavor. These models may provide insight into political discourse, knowledge on how to 'flatten the curve' in the face of a pandemic, and crucial information on the potential impact of human interactions with the environment.

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A Multi-Temperature Kinetic Ising Model Code

```
#Author: Sho Gibbs, Washington and Lee Department of Physics and Engineering
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint
import math
from scipy.optimize import curve_fit

def ODO(x, a, b, c,d,e):
    return a * np.exp(b * x) + c* np.exp(d*x)+e

def ODOPaper(x,i,j,k):
    Tau = math.sqrt(g0[0] * g0[1])
    return i * np.exp(j *(1-Tau)* x) + k* np.exp(j* (1+Tau)*x)

def MatGen(g0,bounds):
    size= g0.size
    A=np.zeros((size,size))
    for i in range(size):
        for j in range(size):
            if i==j:
                A[i,j]=-1
            elif abs(i-j)==1:
                A[i,j]= np.sqrt(g0[i]*g0[j])/2
    if bounds ==0:
        A[0,size-1]=np.sqrt(g0[0]*g0[size-1])/2
        A[size-1,0]=np.sqrt(g0[0]*g0[size-1])/2
    return A

T = (3.0, 2.5, 5, 3.5, 4.5) #modify for number of sub-lattices
N = 435
timestop = 20
ystart = 1
bounds = 1
field = 0
const = 0

    #Cutoff values (modify for number of sub-lattices)
Cutoff1=N/5
Cutoff2=2*N/5
Cutoff3=3*N/5
Cutoff4=4*N/5
    #gamma values based on temprature bath
T0=np.zeros(N)
g0=np.zeros(N)
for i in range(N):
    if i<Cutoff1:
        T0[i]=T[0]
    elif i<Cutoff2:
        T0[i]=T[1]
    elif i<Cutoff3:
```

```
        T0[i]=T[2]
    elif i<Cutoff4:
        T0[i]=T[3]
    else:
        T0[i]=T[4]
    g0[i]= np.tanh(2/T0[i])
    #time array
timesteps= 100
tinc= timestop/ timesteps  #this number is the number of steps
t= np.arange(0,timestop,tinc)
x_data =t

    #set initial spin conditions
y0=np.zeros(N)
for i in range(N):
    if i < Cutoff1:
        y0[i]= 1
    elif i < Cutoff2:
        y0[i] = 1
    elif i < Cutoff3:
        y0[i] = 0
    elif i < Cutoff4:
        y0[i] = -1
    else:
        y0[i] = -1
if ystart==2:
    for i in range(N):
        y0[i]=1

    ##Magnetic field
if field==0:
    b0=np.zeros(N)
if field==1:
    b0=np.zeros(N)+ const
if field==2: #field depending on temperature bath
    b0=np.zeros(N)
    for i in range(N):
        if i < Cutoff1:
            b0[i]= const
        elif i < Cutoff2:
            b0[i] = 0
        else:
            b0[i] = -const
B0=np.zeros(N)
B0=np.tanh(b0/T0)
    #boundary conditons+ solve system
if bounds==1:
    psoln = odeint(f1, y0, t, args=(N,g0,B0))
if bounds==0:
    psoln = odeint(f0, y0, t, args=(N,g0,B0))
magnetization= np.sum(psoln,axis=1)/N
```

```
y_data = magnetization

    ####Matrix work
matrix = MatGen(g0,bounds)
eigVal, eigVec =np.linalg.eig(matrix)
print("eigenvalues:")
print(eigVal)
print("eigenvectors:")
print(eigVec)

xdata = x_data
y = y_data
ydata = y
fig=plt.figure(figsize=(12,12))
plt.plot(xdata, ydata, label='data')
popt, pcov = curve_fit(ODO, xdata, ydata, p0=(-1,-2,1,-1,0), maxfev=100000)
print(T,N)
fitCon=np.round(popt,4)
plt.plot(xdata, ODO(xdata, *popt))
plt.xlabel('x')
plt.ylabel('y')
plt.title((T,field,const), loc='left')
plt.title(fitCon,loc='right')
```

B Social Network Model Code

```
#Author: Daniel Clark, Washington and Lee Department of Physics and Engineering
import networkx as nx
import random
import numpy as np
import pandas as pd


k = 0.00          #Issue value

df = pd.read_excel('Legislature.xlsx', index_col=0)

G = nx.Graph()
H = nx.path_graph(435)
G.add_nodes_from(H)

#Defines a Function to Store the Randomly Generated Starting Votes in a List Object
def old_vote(G):
    starting_votes = []
    for node in G:
        starting_votes.append(df.iat[node, 3])
    return starting_votes

#Defines a Function to Determine the Weight of Interactions Between Representatives
def edge_weight(u, v):
    a1 = df.iat[u, 2]
```

```
    a2 = df.iat[v, 2]
    gamma = np.random.random_sample()
    beta = (df.iat[u, 1])/(df.iat[v, 1])
    w = (beta + gamma*((1-np.abs(a1-a2))))
    wrounded = round(w, 3)
    return wrounded

#Defines a Function to Randomly Create Edges Between Representatives
def add_random_edges(G):
    new_edges = []

    for node in G.nodes():
        connected = [to for (fr, to) in G.edges(node)]
        unconnected = [n for n in G.nodes() if n not in connected and n != node]
        if len(unconnected):
            new = random.choice(unconnected)
            if new != node:
                G.add_edge(node, new, weight=edge_weight(node, new))
                new_edges.append( (node, new) )
                connected.append(new)
    return new_edges

#Defines a Function to Selectively Create Edges Between Representatives of the Same Party
def add_selective_edges(G, P_assort):
    new_edges = []
    for node in G.nodes():
        x = np.rand_random_sample()
        if x > P_assort:
            connected = [to for (fr, to) in G.edges(node)]
            unconnected = [n for n in G.nodes() if n not in connected and n != node
             and df.iat[node,1] == df.iat[n,1]]
            if len(unconnected) != 0:
                new = random.choice(unconnected)
                if new != node:
                    G.add_edge(node, new, weight=edge_weight(node, new))
                    new_edges.append( (node, new) )
                    connected.append(new)
            else:
                for node in G.nodes():
                    connected = [to for (fr, to) in G.edges(node)]
                    unconnected = [n for n in G.nodes() if n not in connected and n != node]
                    if len(unconnected) != 0:
                        new = random.choice(unconnected)
                        if new != node:
                            G.add_edge(node, new, weight=edge_weight(node, new))
                            new_edges.append( (node, new) )
                            connected.append(new)
        else:
            for node in G.nodes():
                connected = [to for (fr, to) in G.edges(node)]
                unconnected = [n for n in G.nodes() if n not in connected and n != node]
```

```
                if len(unconnected):
                    new = random.choice(unconnected)
                    if new != node:
                        G.add_edge(node, new, weight=edge_weight(node, new))
                        new_edges.append( (node, new) )
                        connected.append(new)
    return new_edges

#Defines a Function to Model the First Step of Voting Interactions
def second_vote(G, u, v):  #u and v provide the ideological cutoffs for 'Bloc' voting
    second_votes = []
    for node in G.nodes():
        a1 = df.iat[node, 2]
        connected = [to for (fr, to) in G.edges(node)]
        influence = 0
        starting_vote1 = df.iat[node, 3]
        if df.iat[node,2] < u or df.iat[node,2] > v:
            second_votes.append(starting_vote1)
        else:
            for connection in connected:
                starting_vote2 = df.iat[connection, 3]
                if starting_vote1 == starting_vote2:
                    stay = G.get_edge_data(node, connection, 'weight')
                    hold = stay['weight']
                    influence = influence - hold
                else:
                    sway = G.get_edge_data(node, connection, 'weight')
                    change = sway['weight']
                    influence = influence + change
            if (1-np.abs(a1-k)) < influence:
                if starting_vote1 == 1:
                    second_votes.append(0)
                else:
                    second_votes.append(1)
            else:
                second_votes.append(starting_vote1)
    return second_votes

#Defines a Function to Model Multiple Stages of Interactions
def multiple_vote(G, generations, u, v):
    second_votes = second_vote(G, u, v)
    t = 0
    t_votes = []
    G = nx.Graph()
    H = nx.path_graph(435)
    G.add_nodes_from(H)
    influence = 0
    while t < generations:
        add_random_edges(G)
        for node in G.nodes():
            a1 = df.iat[node, 2]
```

```
            connected = [to for (fr, to) in G.edges(node)]
            starting_vote1 = df.iat[node, 3]
            if df.iat[node,2] < u or df.iat[node,2] > v:
                t_votes.append(starting_vote1)
            else:
                hold = []
                change = []
                if t == 0:
                    previous_vote1 = second_votes[node]
                else:
                        previous_vote1 = t_votes[node+(434*(t-1))]
                for connection in connected:
                    if t == 0:
                        previous_vote2 = second_votes[connection]
                    else:
                        previous_vote2 = t_votes[connection+(434*(t-1))]
                    if previous_vote1 == previous_vote2:
                        stay = G.get_edge_data(node, connection, 'weight')
                        hold.append(stay['weight'])
                    else:
                        sway = G.get_edge_data(node, connection, 'weight')
                        change.append(sway['weight'])
                influence = sum(change) - sum(hold)
                if (1-np.abs(a1-k)) < (influence):
                    if starting_vote1 == 1:
                        t_votes.append(0)
                    else:
                        t_votes.append(1)
                else:
                    t_votes.append(previous_vote1)
        t+=1
        yield t_votes

def multiple_vote_selective(G, generations, u, v):
    second_votes = second_vote(G, u, v)
    t = 0
    t_votes = []
    G = nx.Graph()
    H = nx.path_graph(435)
    G.add_nodes_from(H)
    influence = 0
    while t < generations:
        add_selective_edges(G)
        for node in G.nodes():
            a1 = df.iat[node, 2]
            connected = [to for (fr, to) in G.edges(node)]
            starting_vote1 = df.iat[node, 3]
            if df.iat[node,2] < u or df.iat[node,2] > v:
                t_votes.append(starting_vote1)
            else:
                hold = []
```

```
                change = []
                if t == 0:
                    previous_vote1 = second_votes[node]
                else:
                        previous_vote1 = t_votes[node+(434*(t-1))]
                for connection in connected:
                    if t == 0:
                        previous_vote2 = second_votes[connection]
                    else:
                        previous_vote2 = t_votes[connection+(434*(t-1))]
                    if previous_vote1 == previous_vote2:
                        stay = G.get_edge_data(node, connection, 'weight')
                        hold.append(stay['weight'])
                    else:
                        sway = G.get_edge_data(node, connection, 'weight')
                        change.append(sway['weight'])
                influence = sum(change) - sum(hold)
                if (1-np.abs(a1-k)) < (influence):
                    if starting_vote1 == 1:
                        t_votes.append(0)
                    else:
                        t_votes.append(1)
                else:
                    t_votes.append(previous_vote1)
        t+=1
        yield t_votes

#Look at the Initial Conditions, Create Connections, and Change Votes
first_votes = old_vote(G)
new_edges = add_random_edges(G)
second_votes = second_vote(G, ideological_cutoff_low, ideological_cutoff_high)

#Preform a Multi-Generational Voting Scenario, and Organize the Results
generation_voting = list(multiple_vote(G, generations, cutoffl, cutoffh))
generational_votes = [generation_voting[1]]
final_votes = []
for sublist in generational_votes:
    for item in sublist:
        final_votes.append(item)
seg_length = 435
voting_pattern=[final_votes[x:x+seg_length] for x in range(0,len(final_votes),seg_length)]
pd.DataFrame(voting_pattern).to_excel('houseoutput.xlsx', header=False, index=False)

#Perform a Selective Multi-Generational Voting Scenario
generation_voting = list(multiple_vote_selective(G, generations, cutoffl, cutoffh))
generational_votes = [generation_voting[1]]
final_votes = []
for sublist in generational_votes:
    for item in sublist:
        final_votes.append(item)
seg_length = 435
```

```
voting_pattern=[final_votes[x:x+seg_length] for x in range(0,len(final_votes),seg_length)]
pd.DataFrame(voting_pattern).to_excel('houseoutput.xlsx', header=False, index=False)
```